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Mathematical Reviews

Edited by

R. P. Boas, Jr. E. Hille H. Whitney

J. V. Wehausen, *Executive Editor*

Vol. 14, No. 7 July-August, 1953 pp. 609-712

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MATHEMATICAL REVIEWS

Published monthly, except August, by

THE AMERICAN MATHEMATICAL SOCIETY, Prince and Lemon Streets, Lancaster, Pennsylvania

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Editorial Office

MATHEMATICAL REVIEWS, 80 Waterman St., Providence 6, R. I.

Subscriptions: Price $20 per year ($10 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions may be addressed to MATHEMATICAL REVIEWS, Lancaster, Pennsylvania, but should preferably be addressed to the American Mathematical Society, 80 Waterman St., Providence 6, R. I.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. Its preparation is also supported currently under a contract with the Office of Scientific Research, Headquarters, Air Research and Development Command, U. S. Air Force. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in paragraph (d-2), section 34.40, P. L. and R. of 1948, authorized November 9, 1940.

Mathematical Reviews

Vol. 14, No. 7 JULY–AUGUST, 1953 Pages 609-712

HISTORY

Schmidt, Olaf. On the computation of the ahargana. Centaurus 2, 140–180 (1952).

Brusotti, Luigi. I metodi di esaustione nella storia della matematica. Period. Mat. (4) 30, 241–248 (1952).

Hofmann, Jos. E. Das Problem der Parabel- und Hyperbelquadratur im Wandel der Zeiten. Math.-Phys. Semesterber. 3, 59–79 (1953).

Aleksandrov, P. S. On certain fundamental directions of development of Soviet topology. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk 1952, no. 2, 3–33 (1952). (Russian)

Guzzo, Augusto. Archimede. Filosofia 3, 149–168 (1952).

Starting with a discussion of Euclid's position in Greek mathematics the author describes Archimedes as leading a twofold life, partly devoted to technical invention, partly to theoretical research. The story of his death is interpreted as a symbol of his own appreciation of these two activities: pure science is placed immensely high above technical work. He strove to resemble a classical Greek of the purely idealistic type. Four instincts determining his aims and procedures are mentioned as the dominant factors in his personality as a man of science. *E. J. Dijksterhuis.*

Abellan, Juan. Henry Briggs. Gaceta Mat. (1) 4, 39–41 (1952). (Spanish)

Pajares, Emilio. Bürgi. Gaceta Mat. (1) 4, 157–163 (1 plate) (1952). (Spanish)

Crespo Pereira, Ramón. Georg Cantor. Gaceta Mat. (1) 4, 67–73 (1 plate) (1952). (Spanish)

*Čaplygin, S. A. Sobranie sočinenii. Tom I. Teoretičeskaya mehanika. Matematika. Tom II. Gidrodinamika. Aèrodinamika. Tom III. Matematika i mehanika. Reči i doklady. Tom IV. Kursy lekcii po teoretičeskoi mehanike. [Collected works. Vol. I. Theoretical mechanics. Mathematics. Vol. II. Hydrodynamics. Aerodynamics. Vol. III. Mathematics and mechanics. Speeches and reports. Vol. IV. Courses of lectures on theoretical mechanics.] Gosudarstv. Izdat. Tehn.-Teoret. Lit., Moscow-Leningrad, 1948, 1948, 1950, 1949. 484 pp. (1 plate), 644 pp. (1 plate), 467 pp. (1 plate), 616 pp. (1 plate). 19.50 rubles per volume.

The first three volumes contain notes by the editors on the included papers. Volume 3 also contains essays on Čaplygin's work in mathematics, theoretical mechanics, hydrodynamics, wing and propeller profiles, and the theory of elasticity. There is also a biographical essay by V. Golubev and a complete list of his published works.

Sretenskiĭ, L. N. The scientific production of S. A. Čaplygin. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk 1953, 106–108 (1 plate) (1953). (Russian)

*Ore, Oystein. Cardano, the gambling scholar. With a translation from the Latin of Cardano's "Book on games of chance", by S. H. Gould. Princeton University Press, Princeton, N. J., 1953. xiv+249 pp. $4.00.

Guzzo, Augusto. Euclide. Filosofia 3, 45–82 (1952).

Opposing the current conception of Euclid's Elements as a rigid system in which from indubitable fundamental assumptions necessary conclusions are drawn, the author also finds in the work the procedure of invention: Euclid gradually discovers the propositions which he sets forth in an harmonical but not rigid way. This idea is developed in a discussion of various parts of the work.

E. J. Dijksterhuis (Oisterwijk).

Boyer, Carl B. Fermat and Descartes. Scripta Math. 18 (1952), 189–217 (1953).

Crespo, R. Leopoldo Kronecker. Gaceta Mat. (1) 4, 199–204 (1 plate) (1952). (Spanish)

*Leĭbenzon, L. S. Sobranie trudov. Tom. I. Teoriya uprugosti. [Collected works. Vol. I. Theory of elasticity.] Izdat. Akad. Nauk SSSR, Moscow, 1951. 468 pp. (1 plate). 26 rubles.

In addition to thirteen papers this volume contains the author's book "Variational methods for the solution of problems of the theory of elasticity" [GTTI, Moscow-Leningrad, 1943; these Rev. 6, 79].

*Yanovskaya, S. A. Peredovye idei N. I. Lobacevskogo—orudie bor'by protiv idealizma v matematike. [The leading ideas of N. I. Lobačevskiĭ—a combat weapon against idealism in mathematics.] Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1950. 83 pp.

Chand, Uttam. Obituary: Late Dwarka Nath Nanda. J. Indian Soc. Agric. Statistics 4, 109–112 (1952).

van Heel, A. C. S. Newton's work on geometrical optical aberrations. Nature 171, 305–306 (1953).

Crespo, R. Agustín de Pedrayes. Gaceta Mat. (1) 4, 3–8 (1 plate) (1952). (Spanish)

Crespo Pereira, R. On Pedrayes' problem. Gaceta Mat. (1) 4, 164–174 (1952). (Spanish)

*Riemann, Bernhard. **Gesammelte mathematische Werke und wissenschaftlicher Nachlass.** Dover Publications, Inc., New York, N. Y., 1953. xix+558+viii+116 pp. Clothbound $4.95; paperbound $2.25.

Photo-offset reprint of the 2d edition [Teubner, Leipzig, 1892] and of the Nachträge [Teubner, Leipzig, 1902]. An unpaged introduction of 8 pages was written by Hans Lewy for this edition.

Crespo, R. Th. Skolem. Gaceta Mat. (1) **4**, 109–112 (1 plate) (1952). (Spanish)

Guzzo, Augusto. Tolomeo. Filosofia **3**, 351–370 (1952).

After a discussion of the Archimedean work Arenarius it is explained why it is so difficult for the modern reader to pass an equitable judgment on Ptolemy: unconsciously he is still under the influence of the passionate controversy about the geocentric view of the universe which was carried on during and after the Renaissance and in which Ptolemy was wrongly considered its chief exponent. The author presents a more dispassionate view of the work of the man who by his synthesis of ancient astronomy prepared the evolution of this discipline in later times.

E. J. Dijksterhuis (Oisterwijk).

*Vinogradov, I. M. **Izbrannye trudy.** [Selected works.] Izdat. Akad. Nauk SSSR, Moscow, 1952. 436 pp. (1 plate). 21.70 rubles.

A selection of 26 papers by the author including his long monograph in Trudy Mat. Inst. Steklov. **23**, 1–109 (1947) [these Rev. **10**, 599]. A complete list of Vinogradov's published mathematical work is included.

ALGEBRA

García Pradillo, Julio. Permutations of n elements with i inversions. Gaceta Mat. (1) **4**, 125–130 (1952). (Spanish)

A number of known relations for the number of permutations of n elements with i inversions, say $I_{n,i}$; also the following which may be new

$$I_{n,i}=\sum_{0}^{i}\binom{n-i}{j}I_{i+k,i-j},\quad n\geqq i.$$

J. Riordan (New York, N. Y.).

Yamamoto, Koichi. Note on enumeration of 7×7 Latin squares. Bull. Math. Statist. **5**, 1–8 (1952).

This paper systematizes the enumeration of Latin rectangles. The method is applied to the enumeration of 7×7 Latin squares and Sade's result [Enumération des carrés latins, Marseille, 1948; this Rev. **10**, 278] is verified. The author uses several symbols without proper definition and the reviewer was unable to follow his paper.

H. B. Mann (Columbus, Ohio).

Bose, R. C. A note on Nair's condition for partially balanced incomplete block designs with $k>r$. Calcutta Statist. Assoc. Bull. **4**, 123–126 (1952).

The author gives a purely combinatorial proof of K. R. Nair's necessary condition [Sankhyā **6**, 255–259 (1943); these Rev. **5**, 29] for the existence of a partially balanced incomplete block design with given parameters.

H. B. Mann (Columbus, Ohio).

Roy, Purnendu Mohon. A note on the resolvability of balanced incomplete block designs. Calcutta Statist. Assoc. Bull. **4**, 130–132 (1952).

The author proves the following two theorems. (1) If the number v of varieties of a balanced incomplete block design with parameters v, b, r, k, λ is divisible by k, then $b\geqq v+r-1$. (2) If $v=nk$, $b=nr$ and if there exists a set of n blocks containing a complete replication and if, moreover, $b=v+r-1$, then $k^2\equiv 0 \pmod{v}$. These two theorems generalize results of R. C. Bose [Sankhyā **6**, 105–110 (1942); these Rev. **4**, 237].

H. B. Mann (Columbus, Ohio).

Afriat, S. N. The rank and multiplicity theorem for the reduction of quadratic forms. Math. Gaz. **37**, 27–28 (1953).

Direct and independent proof is given to the theorem, implicit in the possibility, and fundamental for the procedure of the simultaneous canonical reduction of a pair of real quadratic forms, one of which is positive definite.

Extract from the paper.

Mitchell, B. E. Normal and diagonalizable matrices. Amer. Math. Monthly **60**, 94–96 (1953).

Known conditions that a matrix be normal are simply derived and analogous results for diagonalizable matrices are obtained from the theorem: A is diagonalizable if and only if HAH^{-1} is normal for some positive definite hermitian H.

W. Givens (Knoxville, Tenn.).

Schneider, H. Theorems on normal matrices. Quart. J. Math., Oxford Ser. (2) **3**, 241–249 (1952).

Let $A=HU=UH$ and B be normal matrices, H the nonnegative definite and U the unitary factor in the polar factorization of A. Then (1) $AB=BA$ if and only if (*) $\sum_{i=1}^{n}|f(\alpha_i)\beta_i|^2=\sum_{i=1}^{n}|\gamma_i|^2$ for some ordering of the roots α_i and β_i of A and B, where γ_i are the roots of $f(A)B$ and f is any (scalar) polynomial [improvement of a known result by restricting $F(A, B)$ to $f(A)B$]; AB is normal if and only if (*) holds for those polynomials f for which the Hermitian polar matrix of $f(A)$ is a polynomial $g(H)$ in H (then $f(\alpha_i)=f(\alpha_j)=g(|\alpha_i|)$ when $|\alpha_i|=|\alpha_j|$). Misprint: p. 246, line −7, for V read K.

W. Givens (Knoxville, Tenn.).

Lloyd, E. H. The direct product of matrices. Math. Gaz. **37**, 29–33 (1953).

Expository article on the Kronecker product.

J. L. Brenner (Pullman, Wash.).

Schwerdtfeger, Hans. Sur les matrices permutables avec leur dérivée. Univ. e Politecnico Torino. Rend. Sem. Mat. **11**, 329–333 (1952).

If $X(t)$ is a non-derogatory matrix with elements analytic functions of t and commuting with its derivative in a neighborhood of t_0, then $X(t_1)X(t_2)=X(t_2)X(t_1)$ for t_1 and t_2 in the domain of analyticity of $X(t)$. Special three by three matrices with equal roots are also considered. Misprints are: p. 331, line −2: for D read B; p. 332 in [b]: the 2, 2 element is λ.

W. Givens (Knoxville, Tenn.).

Ascoli, Guido. Remarque sur une communication de Mr. H. Schwerdtfeger. Univ. e Politecnico Torino. Rend. Sem. Mat. **11**, 335–336 (1952).

[Cf. the preceding review.] A three by three matrix $X(t)$ of rank one with monomial elements of at most degree four

and such that $XX'=X'X(=0)$, yet $X(t_1)X(t_2)\neq X(t_2)X(t_1)$, is exhibited. *W. Givens* (Knoxville, Tenn.).

Perfect, Hazel. Note on a previous paper on matrices with positive elements. Quart. J. Math., Oxford Ser. (2) **3**, 187–188 (1952).

Let A be a matrix with positive elements, $Ay=\chi y$ with χ the maximal root and M_r and m_r the maximum and minimum of the ratio of corresponding components of $A^r u$ and y for an arbitrary positive vector u. Then M_r/m_r, $r=0, 1, 2, \cdots$, is a steadily decreasing sequence with limit one. For A non-symmetric, a numerical example shows that $(y'A^r u)/(|y|\cdot|A^r u|)$ need not be monotonic increasing although this was proved for symmetric A and conjectured for all positive A in a previous paper [same J. (2) **2**, 286–290 (1951); these Rev. **13**, 425]. *W. Givens.*

Hoffman, A. J., and Wielandt, H. W. The variation of the spectrum of a normal matrix. Duke Math. J. **20**, 37–39 (1953).

If A and B are normal matrices, there exists a suitable numbering of their eigenvalues so that $\sum_i|\alpha_i-\beta_i|^2\leq\|A-B\|^2$, where $\|K\|^2=\sum_{i,j}|k_{ij}|^2$. The proof is obtained by (1) showing by a simple induction on n the known result that the region in n^2-dimensional space consisting of the non-negative matrices with row sum and column sum one is a convex polyhedron with vertices which are permutation matrices, and (2) proving that $\min\|A_0-VB_0V^*\|^2$ for V unitary is attained for V a permutation matrix by minimizing a linear form on the region of (1). Misprints: a star is omitted from a B_0 in the displayed formula on p. 37 and an exponent 1/2 in the definition of $d(\alpha,\beta)$ on p. 39.
W. Givens (Knoxville, Tenn.).

Ostrowski, A. M. Note on bounds for determinants with dominant principal diagonal. Proc. Amer. Math. Soc. **3**, 26–30 (1952).

Let A be the determinant of $(a_{\mu\nu})$ where $\mu, \nu=1, \cdots, n$ and $a_{\mu\mu}\neq 0$ for $\mu=1, \cdots, n$. Set

$$\alpha_{\mu\nu}=|a_{\mu\nu}|,\quad \sum_{\nu=1}^{n}\alpha_{\mu\nu}-\alpha_{\mu\mu}=s_\mu=\sigma_\mu\alpha_{\mu\mu},\quad t_\mu=\max_{\nu\neq\mu}\sigma_\nu.$$

Let A_m denote the minor of a_{mm}. The author's principal result is the following theorem: If $t_m\leq 1$ and if $t_m\sigma_m<1$, then $A=A_m(a_{mm}+\theta_m t_m s_m)$ where $|\theta_m|\leq 1$. This result can be applied repeatedly to obtain generalizations of inequalities proved by Price [same Proc. **2**, 497–502 (1951); these Rev. **12**, 793] and Oeder [Amer. Math. Monthly **58**, 37 (1951)], which were improvements of others obtained still earlier by the author. The proof of the theorem is based on the following inequality: If $(b_{\mu\nu})$ is the inverse of the matrix $(a_{\mu\nu})$, if $\beta_{\mu\nu}=|b_{\mu\nu}|$ for $\mu, \nu=1, \cdots, n$, and if all σ_μ are less than 1, then $\beta_{\mu\nu}\leq\sigma_\mu\beta_{\nu\nu}$ for $\mu\neq\nu$, and $b_{\mu\mu}=(a_{\mu\mu}+\theta_\mu t_\mu s_\mu)^{-1}$ where $|\theta_\mu|\leq 1$. *G. B. Price* (Lawrence, Kan.).

Ballieu, Robert. Sur les rangs-lignes à droite de deux matrices inverses. Ann. Soc. Sci. Bruxelles. Sér. I. **66**, 119–124 (1952).

Let a, b be two elements of a sfield such that $ab\neq ba$, $b^{-1}(a^{-1}b^{-1}-b^{-1}a^{-1})^{-1}(ba^{-1}-a^{-1}b)(ab-ba)^{-1}b^{-1}(ab-ba)\neq 1$. Then

$$\begin{pmatrix}1 & a\\ b & ab\end{pmatrix}$$

has right row-rank equal to 1, whereas the inverse matrix has right row-rank equal to 2. No two such elements a, b can be found in a sfield of quaternions. In a sfield, it is always possible to find a 3×3 matrix with right row-rank not equal to that of its inverse. This fact is established by constructing such matrices. *J. L. Brenner.*

Chamberlin, Eliot, and Wolfe, James. Multiplicative homomorphisms of matrices. Proc. Amer. Math. Soc. **4**, 37–42 (1953).

Let G denote a multiplicative semi-group with identity element e and zero element 0 such that the square roots of e commute with each other. If $\mathfrak{M}_n$ is the ring of all $n\times n$ matrices over a commutative ring R with unit, let $\mathfrak{M}_n^*$ denote the multiplicative system generated by the "elementary" row transformations of the identity matrix (multiplication of rows by singular elements is allowed). For some rings, in particular for Euclidean rings, $\mathfrak{M}_n^*=\mathfrak{M}_n$. It is shown that if $n\neq 2$, for every multiplicative homomorphism Φ of $\mathfrak{M}_n^*$ onto G, there is a unique multiplicative homomorphism ϕ of R into G such that $\Phi(B)=\phi(\det B)$. If $n=2$, the situation is more complicated, but by specializing the nature of G, the authors determine all multiplicative homomorphisms of $\mathfrak{M}_2^*$ onto G. The proofs are straightforward and computational in nature. More generality is maintained in the paper than is indicated in this brief review. *M. Henriksen.*

Charrueau, André. Formules matricielles relatives aux complexes linéaires et aux faisceaux de complexes linéaires. C. R. Acad. Sci. Paris **235**, 860–862 (1952).

The fundamental identity connecting two skew-symmetric matrices of order four and their duals is noted and formulas for the product of n non-singular null-polarities belonging to a pencil are derived. *W. Givens.*

Charrueau, André. Formules matricielles relatives aux complexes linéaires et aux faisceaux de complexes linéaires. II. C. R. Acad. Sci. Paris **235**, 931–933 (1952).

The formulas of the paper reviewed above are now given for an arbitrary choice of two null-polarities as base of the pencil. *W. Givens* (Knoxville, Tenn.).

Charrueau, André. Sur certaines suites et certains cycles de projectivités. C. R. Acad. Sci. Paris **236**, 455–457 (1953).

Some properties of the projectivities in the group generated by two null-polarities in a non-special pencil in projective three-space are noted. [Cf. the two reviews above.]
W. Givens (Knoxville, Tenn.).

Pinl, M. Isotrope Vektoren im erweiterten Hermiteschen Raum. Comment. Math. Helv. **26**, 323–327 (1952).

Das hyperkomplexe System in vier Einheiten mit der Multiplikationstabelle $e_0e_\alpha=e_\alpha e_0=e_\alpha$, $e_1^2=e_2^2=-e_3^2=e_0$, $e_1e_2=-e_2e_1=e_3$, $e_3e_2=-e_2e_3=e_1$, $e_1e_3=-e_3e_1=e_2$ enthält den Körper der komplexen Zahlen $(\alpha e_0+\beta e_3)$. Definiert man $\bar{x}=\xi_0e_0-\sum\xi_ie_i$ als die zu $x=\xi_0e_0+\sum\xi_ie_i$ konjugierte Zahl, so ist $x\bar{x}$ für komplexe Zahlen positiv-definit, nicht aber für hyperkomplexe Zahlen. Diese Erweiterung ermöglicht die Einführung isotroper Vektoren in Bezug auf den Fundamentaltensor $\sum x_i\bar{x}_i$. Auch ein zweites hyperkomplexes System mit der Tabelle $(e_0e_\alpha=e_\alpha e_0=e_0,\ e_1^2=-e_0,\ e_2^2=e_3^2=e_2e_3=e_3e_2=0, e_1e_2=-e_2e_1=e_3, e_3e_1=-e_1e_3=e_2)$ kann dazu angewendet werden. *J. Haantjes* (Leiden).

Kurepa, Gjuro. Rational numbers as ordered triplets of natural integers. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **7**, 133–139 (1952). (Serbo-Croatian summary)

The ordered rational field is constructed from ordered triples of positive integers; see also Kurepa, C. R. Acad. Sci. Paris **231**, 316–318 (1950) [these Rev. **12**, 154].

G. Birkhoff (Cambridge, Mass.).

Abstract Algebra

***Haupt, Otto. Einführung in die Algebra. Erster Teil.** 2te Aufl. Akademische Verlagsgesellschaft, Geest & Portig K.-G., Leipzig, 1952. xviii+370 pp.

According to the preface this edition differs from the first edition [1929] only in the correction of known errors, additional references, and some small improvements.

Petresco, Julian. Théorie relative des chaines. III. Isoconformisme. C. R. Acad. Sci. Paris **236**, 651–653 (1953).

Continuing this series of notes [same C. R. **235**, 226–228, 1087–1089 (1952); these Rev. **13**, 901; **14**, 346], two ascending chains $\{A_i\}$ and $\{B_j\}$ are called $0-\cap$-isoconformal if $A_{i,j-1}\leqq X\leqq A_{i,j}$ and $B_{j,i-1}\leqq Y\leqq B_{j,i}$, where

$$A_{i,j}=A_{i-1}\cup(A_i\cap B_j) \quad \text{and} \quad B_{j,i}=B_{j-1}\cup(B_j\cap A_i)$$

[misprinted in first paper and review cited], imply

$$A_{i-1}\cup\{A_i\cap[B_{j-1}\cup(B_j\cap X)]\}=X,$$
$$B_{j-1}\cup\{B_j\cap[A_{i-1}\cup(A_i\cap Y)]\}=Y$$

for all i, j. Combining this (and generalizations) with previous notions leads to theorems of the Schreier-Jordan-Zassenhaus type. *P. M. Whitman* (Silver Spring, Md.).

Sorkin, Yu. I. Independent systems of axioms defining a lattice. Ukrain. Mat. Žurnal **3**, 85–97 (1951). (Russian)

Detailed investigation of minimal sets of axioms for a lattice. *P. M. Whitman* (Silver Spring, Md.).

Bergmann, Gustav. Multiplicative closures. Portugaliae Math. **11**, 169–172 (1952).

Every multiplicative closure operator f in a lattice, satisfying $x\subset fx=ffx$, $f(x\cap y)=fx\cap fy$, determines a congruence $fx=fy$. Under the ascending chain condition, every congruence determines $fx=\max[y|y\equiv x]$. In a lattice with 0 in which all intervals $[0, x]$ are relatively complemented, $fx=x\cup f0$. *R. C. Lyndon* (Princeton, N. J.).

Raney, George N. Completely distributive complete lattices. Proc. Amer. Math. Soc. **3**, 677–680 (1952).

Let L be a complete lattice. The complete distributivity (c.d.) of L is defined by two dual relations [cf. G. Birkhoff, Lattice theory, rev. ed., Amer. Math. Soc. Colloq. Publ., v. 25, New York, 1948, p. 166; these Rev. **10**, 673]. The principal result is given by the following theorem. (I) Each of these relations implies the c.d. of L. (II) The c.d. of L is a necessary and sufficient condition for L to be a complete homomorphic (not necessarily isomorphic) image [loc. cit. p. 50] of a complete ring of sets (i.e., a family R of subsets of a set containing with every $F\subset R$ the intersection and the union of F). An element $x\in L$ is called completely join-irreducible if for every $K\subset L$ such that $x\leqq\bigcup K$ there exists a $y\in K$ containing x. The author proves that for L to be isomorphic with a complete ring of sets it is necessary and sufficient that every element of L be the join of a set of completely join-irreducible elements.

O. Borůvka and *M. Novotný* (Brno).

Schützenberger, Marcel Paul. Construction du treillis modulaire engendré par deux éléments et une chaîne finie discrète. C. R. Acad. Sci. Paris **235**, 926–928 (1952).

The free (self-dual) modular lattice L generated by 1+1+n (two unordered elements and a chain) is described; it contains at most $3n^2$ join-irreducible elements. This essentially solves Problem 29 of the reviewer's "Lattice Theory" [Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948; these Rev. **10**, 673]. Using his own earlier results [same C. R. **218**, 818–819 (1944); **221**, 218–220 (1945); these Rev. **7**, 409, 235], the author then obtains a simple necessary and sufficient condition that a homomorphic image of L be distributive. Proofs are only outlined.

G. Birkhoff (Cambridge, Mass.).

Szász, G. On the structure of semi-modular lattices of infinite length. Acta Sci. Math. Szeged **14**, 239–245 (1952).

Let L be a lattice which is semi-modular, in the sense that x covers $x\cap y$ implies $x\cup y$ covers y; finite chain conditions are not assumed. Theorem: The Jordan-Dedekind condition holds in the sense that if there is any finite maximal chain between a and b $(a<b)$, then all maximal chains between them are finite and of the same length. If L is semi-modular of infinite length, with 0 and I, and if for each $a\neq I$ there is a finite maximal chain between 0 and a, then no element (except 0, I) has a complement. If $0\in L$ and $a\in L$, then x is called a semi-complement of a if $a\cap x=0$, $x\neq 0$. Theorems: If L is semi-modular of infinite length, with 0 and I, the infinite distributive laws hold, and for some $a\neq I$ there is a finite maximal chain between 0 and a, then a has a semi-complement x such that all maximal chains (if any) between 0 and x are infinite. If L is semi-modular and semi-complemented, with 0 and I, and has only a finite number r of points, and each $a\in L$ $(a\neq 0)$ contains at least one point, then the length of L does not exceed r. *P. M. Whitman.*

Fuchs, L. On subdirect unions. I. Acta Math. Acad. Sci. Hungar. **3**, 103–120 (1952). (Russian summary)

Let A and B be two algebraic structures (in the sense of Bourbaki, i.e., with binary operations only) of the same type (i.e., with the same operations, rules, and, possibly, operator domains). The author wishes to construct all structures G which are subdirect unions of A and B. For this purpose it is obviously sufficient that some homomorphic image of A is isomorphic to some homomorphic image of B. For then G can be taken as the set of all those pairs (a, b) with $a\in A$, $b\in B$ in which a and b belong to residue-classes (modulo the respective kernels of the homomorphisms) which correspond to each other in the given isomorphism and where the operations on pairs are performed component-wise. The author shows that in the case of groups, rings, Boolean algebras, etc. (more generally in all structures in which homomorphisms are uniquely determined by the set of elements mapped on the neutral element) this sufficient condition is also necessary: one obtains all subdirect unions of A and B by choosing suitable substructures A_0 and B_0 with a definite isomorphism $A/A_0\cong B/B_0$. Simple examples, such as cyclic or quasi-cyclic p-groups, follow the general discussion. *K. A. Hirsch.*

Jaffard, P. Théorie arithmétique des anneaux du type de Dedekind. Bull. Soc. Math. France **80**, 61–100 (1952).

A ring of Dedekind type is defined to be a commutative ring A with unit in which every integral ideal $\mathfrak{a}$ can be decomposed into a product $\mathfrak{a}=\mathfrak{a}_1\mathfrak{a}_2\cdots\mathfrak{a}_n$ of ideals $\mathfrak{a}_i$, each of which is contained in exactly one maximal ideal $\mathfrak{m}_i$ of A. Then such a decomposition is possible with $\mathfrak{m}_i\neq\mathfrak{m}_j$ if $i\neq j$, and the $\mathfrak{m}_i$ and $\mathfrak{a}_i$ are uniquely determined by $\mathfrak{a}$. $\mathfrak{a}_i$ is called the component of $\mathfrak{a}$ relative to $\mathfrak{m}_i$. A Dedekind ring (=Z.P.I.-Ring) is a ring of Dedekind type. It is shown that a ring A is of Dedekind type if and only if (1) the intersection of any infinite set of distinct maximal ideals of A is the zero ideal, and (2) each prime ideal of A is contained in exactly one maximal ideal of A.

Let $\mathfrak{o}$ be an integral domain, and K its field of quotients. Two valuations v_1, v_2 of K are said to be independent on $\mathfrak{o}$ if, for every couple (ξ_1, ξ_2) of non-negative elements ξ_i of the value groups Γ_i of v_i $(i=1, 2)$, there exists an element a of $\mathfrak{o}$ such that $v_i(a)=\xi_i$ $(i=1, 2)$. A family $\mathfrak{V}=(v_\iota)_{\iota\in I}$ of valuations of K is called a family of definition of $\mathfrak{o}$ if, for all $x\in K$, $v_\iota(x)\geqq 0$ (all $\iota\in I$) if and only if $x\in\mathfrak{o}$. The family $\mathfrak{V}$ is called regular if, for every $x\in K$, $v_\iota(x)\neq 0$ for only a finite set of indices ι. Let $\mathfrak{o}$ be a multiplication ring (every finite ideal invertible) of Dedekind type. Let $\mathfrak{M}=(\mathfrak{m}_\iota)_{\iota\in I}$ be the set of maximal ideals of $\mathfrak{o}$. $\mathfrak{o}_{\mathfrak{m}_\iota}$ is a valuation ring [Krull, Idealtheorie, Springer, Berlin, 1935, p. 111]; let v_ι be the corresponding valuation of K, and Γ_ι the value group of v_ι. Then the "canonical" family $\mathfrak{V}=(v_\iota)_{\iota\in I}$ is a regular family of definition of $\mathfrak{o}$, and the v_ι are independent in pairs. The lattice-ordered semigroup of ideals of $\mathfrak{o}$ is shown to be isomorphic to that consisting of all systems $(\Omega_\iota)_{\iota\in I}$ of upper-classes Ω_ι of the groups Γ_ι such that $\Omega_\iota=\Gamma_{\iota+}$ for all but a finite set of indices ι. The ideal $\mathfrak{a}$ corresponding to $(\Omega_\iota)_{\iota\in I}$ consists of all $x\in K$ such that $v_\iota(x)\in\Omega_\iota$ (all $\iota\in I$). The fact that $\mathfrak{a}$ is uniquely defined thereby, depends on a theorem of Lorenzen [Math. Z. **45**, 533–553 (1939), Satz 15; these Rev. 1, 101].

If, in addition, $\mathfrak{o}$ is "einartig" (every prime ideal $\neq(0)$ is maximal), then the value groups Γ_ι are archimedean, and the two main questions on divisibility of ideals can be completely answered: (1) $\mathfrak{b}$ divides $\mathfrak{a}$ (i.e., there exists an ideal $\mathfrak{c}$ of $\mathfrak{o}$, not necessarily integral, such that $\mathfrak{a}=\mathfrak{b}\mathfrak{c}$) if and only if, for every maximal ideal of $\mathfrak{o}$ relative to which the component of $\mathfrak{a}$ is finite, the component of $\mathfrak{b}$ is likewise finite; (2) given that $\mathfrak{a}=\mathfrak{b}\mathfrak{c}$, the number of possible solutions $\mathfrak{c}'$ of $\mathfrak{a}=\mathfrak{b}\mathfrak{c}'$ is finite, and $\mathfrak{c}$ is unique if and only if $\mathfrak{b}$ divides $\mathfrak{a}:\mathfrak{b}$. The multiplicative semigroup of ideals of $\mathfrak{o}$ is the class sum of groups; the largest subgroup $\mathfrak{G}(\mathfrak{a})$ thereof containing $\mathfrak{a}$ consists of all ideals $\mathfrak{b}$ dividing and divisible by $\mathfrak{a}$.

In three earlier notes, the author introduced the notion of "filet" in a lattice-ordered abelian group [C. R. Acad. Sci. Paris **230**, 1024–1025, 1125–1126, 1631–1632 (1950); these Rev. **11**, 579, 640]. The same notion, now called a "stria" (une strie), is introduced in the lattice of ideals of a commutative ring with unit in the first paragraph of the present paper, and in general partially ordered sets in the concluding paragraph. [Note: the reviewer questions the corollary on p. 96.] *A. H. Clifford* (Baltimore, Md.).

Amitsur, S. A. The identities of PI-rings. Proc. Amer. Math. Soc. **4**, 27–34 (1953).

This paper inaugurates the program of studying a ring A by means of its universal ring U, the free ring in an infinite number of indeterminates reduced modulo all the identities which hold in A. It is proved that the (Jacobson) radical of U is its maximal nil ideal, and that U is semi-simple if A has no nilpotent ideals. Among the interesting consequences are the following two. If A satisfies any identity, it satisfies one of the form $S_{2n}=0$, where S_n is the alternating sum over the $n!$ products of n variables. A can be embedded in an n by n matrix ring over a commutative ring if and only if it satisfies all identities of the n by n total matrix algebra.

I. Kaplansky (Chicago, Ill.).

McCoy, Neal H. Factorization of certain polynomials over a commutative ring. Duke Math. J. **20**, 113–118 (1953).

Let R denote any commutative ring with unit element 1, I the ring of integers. Let $R[x_1, \cdots, x_n]=R[x]$ be the ring of polynomials in n indeterminates with coefficients in R. An element f of $R[x]$ is called primitive if the ideal generated by the coefficients of the terms of f of positive degree is R. After noting that rings exist in which no primitive polynomial is irreducible, the author defines an element f of $R[x]$ to be weakly irreducible if for each factorization $f=gh$, where g and h are non-units of $R[x]$, there exist non-zero idempotents e_1, e_2 of R with $e_1+e_2=1$, and elements c_i, d_i of $e_iR[x]$ with $c_id_i=e_i$ $(i=1, 2)$, such that $g=c_1+c_2f$ and $h=d_1f+d_2$.

Theorem 1 gives an intricate sufficient condition that a primitive f of $R[x]$ be weakly irreducible. Theorem 1 easily yields Theorem 2: If an element f of $I[x]$ is irreducible over every field, then f is weakly irreducible over every ring. Theorem 3: If an element f of $I[x]$ is irreducible over every field whose characteristic is not one of a finite set K of primes, then f is weakly irreducible over every ring none of whose elements have additive order a prime of K.

The conclusion of Theorem 2 was previously obtained by Mařík [Acta Fac. Nat. Univ. Carol., Prague no. **191** (1949); these Rev. **12**, 76] for the special case when f is a determinant with indeterminate elements.

M. Henriksen (Lafayette, Ind.).

Kneser, Martin. Bemerkung über die Primpolynomzerlegung in endlich vielen Schritten. Math. Z. **57**, 238–240 (1953).

Let k be a field, K an extension of k of finite degree. It is well known since Kronecker that there exists a method for determining in a finite number of steps the irreducible factors of polynomials in $K[T]$ provided such a method exists for $k[t]$ and K/k is separable. The author shows by example that the separability condition may not be omitted.

E. R. Kolchin (New York, N. Y.).

Herstein, I. N. A generalization of a theorem of Jacobson. III. Amer. J. Math. **75**, 105–111 (1953).

Theorem: If R is a ring with center Z such that every x in R satisfies $x^{n(x)}-x$ in Z, where $n(x)>1$ is an integer depending on x, then R is commutative. This result was conjectured by the author in an earlier paper (in which $n(x)$ was assumed to be a fixed integer) [same J. **73**, 756–762 (1951); these Rev. **13**, 426]. The proof given here reduces the theorem to the previous case ($n(x)$ fixed) by means of the Jacobson structure theorems for rings and a recent field-theoretic result of Krasner (see the following review).

R. D. Schafer (Philadelphia, Pa.).

Krasner, Marc. The non-existence of certain extensions. Amer. J. Math. **75**, 112–116 (1953).

Theorem: If K is a finite extension of a field k such that every α in K satisfies an equation $x^{n(\alpha)}-x-a_\alpha=0$, where

$n(\alpha)>1$ depends on α and $a_n \in k$ also depends on α, then k is an algebraic extension of a finite field. This theorem is used by Herstein in the paper reviewed above to generalize a theorem of Jacobson on rings. The proof assumes that k is not an algebraic extension of a finite field. Then some K satisfying the hypotheses of the theorem is one of the following: (a) a pure transcendental extension $R(t)$ of the rational number field R; (b) a pure transcendental extension $P(t)$ of the field P of p elements; (c) an algebraic number field of finite degree. Each of these possibilities is eliminated by the methods of valuation theory.

R. D. Schafer (Philadelphia, Pa.).

Ikeda, Masatoshi. On a theorem of Kaplansky. Osaka Math. J. **4**, 235–240 (1952).

The theorem in question is the following: if D is a division ring with center Z and every element satisfies $x^{n(x)} \in Z$, then D is commutative. The author generalizes this single power of x to a polynomial $\sum c_i x^{n_i}$ with fixed coefficients in the prime subfield of D and variable exponents $n_0>n_1>\cdots$ subject to the condition that n_1 remains bounded. This generalization also covers the case $x^{n(x)}-x$ proved by Krasner [see the preceding review]. As in the reviewer's paper [Canadian J. Math. **3**, 290–292 (1951); these Rev. **13**, 101], the task is to prove the impossibility of a certain type of field extension. As compared with the proof of Krasner, who used valuation theory and algebraic number theory, the proof is elementary but elaborate. *I. Kaplansky.*

Shoda, Kenjiro. Zur Theorie der algebraischen Erweiterungen. Osaka Math. J. **4**, 133–143 (1952).

The present paper is a continuation of the author's series of publications on general algebraic systems [for references see these Rev. **11**, 308; **12**, 313]. Here the author generalizes the algebraic extension theory of fields in the sense of Steinitz and also the extension theory of Abelian groups given by Baer and Szele. An element is transcendental over a system $\mathfrak{A}$ when the combined system $\mathfrak{A}(\alpha)$ is the "free" composition of $\mathfrak{A}$ and α. Otherwise α is formally algebraic. The definition of algebraic element requires more strongly a certain congruence condition: there shall exist no normal subsystem $\mathfrak{B}$ of $\mathfrak{A}(\alpha)$ such that $\mathfrak{A}\cap\mathfrak{B}=0$. Among the principal results should be noted the existence of algebraically closed extensions of a system and a decomposition theorem for generalized polynomials. *O. Ore.*

Rédei, L. Die Verallgemeinerung der Schreierschen Erweiterungstheorie. Acta Sci. Math. Szeged **14**, 252–273 (1952).

If Σ and S are algebraic systems of the same kind (semigroups, groups, semirings, or rings), the Schreier extension problem is to characterize all algebraic systems $\mathfrak{S}$ such that $\mathfrak{S}/\Sigma\cong S$. The author obtains results for these four algebraic systems analogous to the following results for semigroups. If S and Σ are semigroups with identity elements e and ϵ respectively, then a Schreier product $S\circ\Sigma$ of S and Σ is an algebraic system consisting of the product set $S\times\Sigma$ with an operation defined by $(a,\alpha)(b,\beta)=(ab, a^b\alpha^b\beta)$, where a^b, α^b ($\in\Sigma$) are any functions of their arguments satisfying the initial conditions $a^e=a$, $e^a=e$, $\alpha^\epsilon=\alpha$, $\epsilon^\alpha=\epsilon$. $S\circ\Sigma$ is a semigroup if and only if (1) $(\alpha\beta)^c=\alpha^c\beta^c$, (2) $a^{b\alpha}b^c=b^c(a^b)^c$, (3) $a^{b\alpha}b^c=(ab)^c(a^b)^c$ hold. All semigroups $\mathfrak{S}$ such that $\mathfrak{S}/\Sigma\cong S$ are Schreier products $S\circ\Sigma$. *R. E. Johnson.*

Kleinfeld, Erwin. On simple alternative rings. Amer. J. Math. **75**, 98–104 (1953).

Let R be a simple alternative, but not associative, ring with center C. The theorems of this paper have been superseded by the author's announced result that, if R is not its own Jacobson radical, then R is a Cayley-Dickson (C-D) algebra over C. It is shown in this paper, for R of characteristic not 2, that (1) R is a C-D algebra over C if and only if there exists an element x in R, but not in C, such that x^2 is a non-zero element in C; (2) if $ab=ba$ implies $(ab)c=a(bc)$ for every c in R, then R is a C-D division algebra over C; (3) R is a C-D algebra if and only if there exist pairwise anti-commutative elements a, b, c in R such that the associator $(ab)c-a(bc)$ is not a divisor of zero.

R. D. Schafer (Philadelphia, Pa.).

Burger, E. Über die Einzigkeit der Cayley-Zahlen. Bemerkung zu einer Arbeit von L. A. Skorniakov. Arch. Math. **3**, 298–302 (1952).

This paper presents a modification of the proofs by Bruck and Kleinfeld [Proc. Amer. Math. Soc. **2**, 878–890 (1951); these Rev. **13**, 526], Kleinfeld [Proc. Nat. Acad. Sci. U. S. A. **37**, 818–820 (1951); these Rev. **13**, 527], and by Skornyakov [Ukrain. Mat. Žurnal **2**, 70–85 (1950); these Rev. **12**, 668; **13**, 1139] that an alternative division ring is either associative or is a Cayley algebra. This consists mainly in showing that having found a Cayley algebra, the elements permuting with this algebra form a commutative field.

Marshall Hall (Columbus, Ohio).

Zassenhaus, Hans J. A group-theoretic proof of a theorem of Maclagan-Wedderburn. Proc. Glasgow Math. Assoc. **1**, 53–63 (1952).

This paper contains a complete proof that every finite skew-field is a field (Wedderburn's Theorem). The proof hinges upon the theorem: If, in a finite group G, the normaliser of every Abelian subgroup coincides with the centraliser of the subgroup, then the group G is Abelian. The paper includes an analysis of Galois fields and their automorphism groups. *F. Kiokemeister.*

Kasch, Friedrich. Zur Erzeugung separabler Algebren. Norsk Mat. Tidsskr. **34**, 97–99 (1952).

A. A. Albert has proved [Bull. Amer. Math. Soc. **50**, 786–788 (1944); these Rev. **6**, 115] that any separable algebra over an infinite field is generated by two elements. In a recent paper [J. Reine Angew. Math. **189**, 150–159 (1951); these Rev. **14**, 239] the author stated that Albert's theorem may be sharpened as follows: the two generators may be taken to be conjugate under an inner automorphism. This note contains the proof of his statement. *R. D. Schafer.*

Albert, A. A. On commutative power-associative algebras of degree two. Trans. Amer. Math. Soc. **74**, 323–343 (1953).

Let A be a central simple commutative power-associative algebra over a field F of characteristic prime to 30. The degree of A is the (unique) number t of u_i in the expression of the unity element $1=u_1+\cdots+u_t$ as a sum of pairwise orthogonal primitive idempotents u_i in A_K, K the algebraic closure of F. The author has previously shown that, if $t>2$, then A is a classical Jordan algebra [same Trans. **69**, 503–527 (1950); these Rev. **12**, 475]. This paper is concerned with algebras of degree two.

Let u be an idempotent of A. Then A is called u-stable if $A_u(\lambda)A_u(1/2)\subseteq A_u(1/2)$ for $\lambda=0, 1$, where $A_u(\alpha)$ consists

of all x in A such that $ux=\alpha x$; if A is u-stable for every idempotent u of A, then A is called stable. The principal theorem of the first portion of this paper states that, if $t>1$, then A is stable if and only if A is a Jordan algebra. The reduction to the case $t=2$ and F algebraically closed is immediate. An important tool theorem in this case is that, if A is merely u-stable for an idempotent $u\neq 1$, then the algebras $A_u(1)$ and $A_u(0)$ are isomorphic.

L. Kokoris has recently constructed a class of central simple commutative power-associative algebras S over F of characteristic p, which are not Jordan algebras, but which are u-stable for a particular idempotent u, and have degree two and dimension $4p$ over F [Proc. Nat. Acad. Sci. U. S. A. **38**, 534–537 (1952); these Rev. **14**, 11]. Each algebra S contains a central simple subalgebra of dimension $3p$. In the second portion of this paper the author proves that the derivation algebras of S and T are isomorphic, each being the direct sum of the three-dimensional central simple Lie algebra of two-rowed matrices of trace zero and the p-dimensional simple Lie algebra of Witt (this latter algebra being the derivation algebra of $B=F[1, x]$, the p-dimensional associative algebra of all polynomials in x with $x^p=0$). By use of the p^r-dimensional commutative associative algebra B_r of all polynomials in $x_1, \cdots, x_r$ with $x_i^p=0$, and a certain set D_m of derivations of B_r, Kokoris' construction is generalized to give new central simple commutative power-associative algebras $S(B_r, D_m)$ and $T(B_r, D_m)$ of dimensions $p^r(m+3)$ and $p^r(m+2)$ respectively, the algebras S and T being the case $r=m=1$. The proof that these new algebras are simple is made to depend on the fact that B_r is D-simple (that is, the only characteristic ideals of B_r are 0 and B_r).

R. D. Schafer (Philadelphia, Pa.).

Nakayama, Tadasi. Note on Galois cohomology. Nagoya Math. J. **5**, 97–104 (1953).

Operations on cochains for group cohomology are considered which involve products of their values as certain of the arguments range over the whole finite group. In particular, for Galois group cohomology of the Galois group G of a field extension K/k, certain such product operations are shown to associate with every $(2m+1)$-dimensional cohomology class for G in the multiplicative group K^* of K a homomorphism of G^{m-1} into the 2-dimensional cohomology group for G in the norm class group $k^*/N_{K/k}(K^*)$. These correspondences are homomorphisms of the $(2m+1)$-dimensional cohomology group for G in K^* into the relevant group of homomorphisms. Similarly, the author obtains a homomorphism of the $2m$-dimensional cohomology group for G in K^* into the group of homomorphisms of G^m into $k^*/N_{K/k}(K^*)$. These are generalizations of the well-known earlier constructions of the author for the case $m=1$. Finally, relative to invariant subgroups of G, the Witt-Akizuki formula (for 2-cocycles) is generalized to arbitrary dimension ≥ 2. Reviewer's note: It appears from recent (unpublished) work of J. Tate that the author's constructions are repeated cochain-chain multiplications which induce Tate's cup product of a cohomology class for G in K^* by a 1-dimensional homology class for G in the additive group Z of the integers. The 1-dimensional homology group for G in Z is isomorphic with G/G', where G' is the commutator subgroup of G. This is the natural setting in which the above results become transparent.

G. Hochschild.

Gurevič, G. B. Quadratic forms in fields of characteristic 2. Trudy Sem. Vektor. Tenzor. Analizu **7**, 227–232 (1949). (Russian)

This paper shows that in the exceptional case of quadratic forms with coefficients in a field of characteristic 2, every form may be reduced by linear transformation to

$$x^1x^2+x^3x^4+\cdots+x^{2\sigma-1}x^{2\sigma}+\omega(x^{2\sigma+1}, \cdots, x^r),$$

where r is the rank of the form, and σ an invariant called the characteristic of the form. In a complete field, any definite quadratic form is of rank ≤ 2, so that in this case, the canonical forms in general are:

I) $x^1x^2+x^3x^4+\cdots+x^{2\sigma-1}x^{2\sigma}$;

II) $(x^1)^2+x^2x^3+x^4x^5+\cdots+x^{2\sigma}x^{2\sigma+1}$;

III) $(x^1)^2+x^1x^2+(x^2)^2+x^3x^4+x^5x^6+\cdots+x^{2\sigma+1}x^{2\sigma+2}$.

L. C. Hutchinson (Boston, Mass.).

Witt, Ernst. Über einen Satz von Ostrowski. Arch. Math. **3**, 334 (1952).

The following proof of Ostrowski's theorem that the only complete Archimedean-valued fields F are the reals R or the complexes $R(i)$ [see Acta Math. **41**, 271–284 (1917)] is based on: the known fact that $F\supset R$; the assertion (unproved) that the differential equation in F: $x^{-1}dx=dy$ provides an isomorphism between the multiplicative group F^* of nonzero elements of F and the additive group F_+ of F if the dimension $[F:R]>2$, i.e., if $F\neq R, R(i)$, since F^* is then simply connected; the immediate consequence that such an isomorphism leads to a contradiction, since F^* contains elements of order 2 (e.g., -1) while F_+ does not, and that therefore $F=R$ or $R(i)$. *G. K. Kalisch.*

Waelbroeck, L. Sur les surcorps du corps des nombres réels. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 1026–1029 (1952).

Let A be a topological division algebra over the reals (continuity of the inverse is tacitly assumed). The Gelfand-Mazur theorem asserts that if A is normed, it is the reals, complexes, or quaternions. Arens [Bull. Amer. Math. Soc. **53**, 623–630 (1947); these Rev. **9**, 6] weakened this hypothesis to local convexity. The reviewer [same Bull. **54**, 809–826 (1948); these Rev. **10**, 179] noted that a complete set of continuous functionals would suffice. The author finds it enough to assume one non-zero functional. (But this immediately yields a complete set by taking the functionals given by $g(x)=f(ax)$.) *I. Kaplansky* (Chicago, Ill.).

Theory of Groups

Popova, Hélène. Logarithmétiques réductibles de quasi-groupes. C. R. Acad. Sci. Paris **235**, 1589–1591 (1952).

The elements of a quasigroup Q of order n are denoted $1, 2, \cdots, n$; n_i is the order of the element i (that is, of the subquasigroup which it generates); N is the order of the logarithmetic L_Q. A given vector $\{j, \cdots, k\}$ with components in Q is said to generate the set L', of order $N'\leq n_j\cdots n_k$, which consists of all distinct vectors of the form $\{j^x, \cdots, k^x\}$, x indicating a (nonassociative) power. L_Q can be represented as the set of vectors generated by $\{1, 2, \cdots, n\}$, with appropriately defined addition and multiplication. Then L_Q is called reducible if there exists a vector $\{j, \cdots, k\}$ for which $N'<n_j\cdots n_k$, by which it is implied that the representation of L_Q can then and only then be simplified by the omis-

sion of redundant components. $N=n_1\cdots n_s$ when L_Q is irreducible.

For a plain quasigroup Q [quasi-groupe uni: see Popova, same C. R. **235**, 1360–1362 (1952); these Rev. **14**, 444] all $n_i=n$. A necessary and sufficient condition for irreducibility in this case is given. The representation can be "reduced" in an obvious way if the range r (defined in the paper quoted) is greater than 1; then $N=n/r$ if the resulting representation is irreducible. Some further results are stated for plain quasigroups with $r=1$. *I. M. H. Etherington.*

Thierrin, Gabriel. Sur quelques classes de demi-groupes. C. R. Acad. Sci. Paris **236**, 33–35 (1953).

The author continues his study of "homogroups" [same C. R. **234**, 1519–1521 (1952); these Rev. **13**, 902]. In the terminology of the reviewer and D. D. Miller [Amer. J. Math. **70**, 117–125 (1948); these Rev. **9**, 330], a homogroup is a semigroup having zeroid elements; no reference, however, is made to this paper. In the terminology of Dubreil [Mém. Acad. Sci. Inst. France (2) **63**, no. 3 (1941); these Rev. **8**, 15], if S is a right superior, neat (net), and symmetric sub-demi-group of a demi-group D, and $R={}_SR=R_S$ is the principal equivalence associated with S, then D/R is a homogroup homomorphic to D. *A. H. Clifford.*

Thierrin, Gabriel. Sur quelques équivalences dans les demi-groupes. C. R. Acad. Sci. Paris **236**, 565–567 (1953).

If H is a subset of a demi-group D, and $a\in D$, the right quotient Q_a of H by a is defined to be the set of elements x of D such that $ax\in H$. The set H is called bound on the right (relié à droite) if the relations $Q_a\cap Q_b\neq 0$, $Q_b\cap Q_c\neq 0$ imply $Q_a\cap Q_c\neq 0$. Then the relation $R_{(H)}$ defined by $aR_{(H)}b$ if and only if $Q_a=Q_b=0$ or $Q_a\cap Q_b\neq 0$ is an equivalence relation such that $R_H\subseteq R_{(H)}$, where R_H is the "principal equivalence" (aR_Hb if $Q_a=Q_b$) defined by P. Dubreil [Mém. Acad. Sci. Inst. France (2) **63**, no. 3 (1941); these Rev. **8**, 15]. For arbitrary H, $R_{[H]}$ is defined to be the transitive extension of $R_{(H)}$. Properties of these two relations are discussed under various further assumptions. No proofs are given.

A. H. Clifford (Baltimore, Md.).

Tamura, Takayuki. On the system of semigroup operations defined in a set. J. Gakugei Coll. Tokushima Univ. **2**, 1–12 (1952).

Let $\mathfrak{U}$ be the set of associative binary operations λ on a set E such that $E\lambda E=E$. For $\lambda,\mu\in\mathfrak{U}$ define $\lambda\gtrsim\mu$ to mean that $(a\lambda b)\mu c=a\lambda(b\mu c)$ for all $a, b, c\in E$. Then $\gtrsim$ is a quasi-ordering of $\mathfrak{U}$. Let S be a one-to-one mapping $x\to x^S$ of E onto itself. For $\lambda\in\mathfrak{U}$ define λ^S by $a\lambda^S b=(a^S\lambda b^S)^{S-1}$. Then $\lambda^S\in\mathfrak{U}$ and $\lambda^S=\lambda$ if and only if S is an automorphism of $E(\lambda)$. If $\lambda\gtrsim\mu$, then $\lambda^S\gtrsim\mu^S$. A group $\mathfrak{G}$ of one-to-one mappings S of E induces a group $\overline{\mathfrak{G}}$ of one-to-one mappings $\lambda\to\lambda^S$ of $\mathfrak{U}$ onto itself, and $\mathfrak{G}\sim\overline{\mathfrak{G}}$. Relations between the quasi-ordering $\gtrsim$ of $\mathfrak{U}$ and the group $\overline{\mathfrak{G}}$ are studied, particularly for finite $\overline{\mathfrak{G}}$. Conditions for $\lambda\gtrsim\mu$ are also given in terms of the regular representations of $E(\lambda)$ and $E(\mu)$. *A. H. Clifford.*

Linés Escardó, E., and Mallol Balmaña, R. On l-groups. Revista Mat. Hisp.-Amer. (4) **12**, 129–136 (1952). (Spanish)

Five sets of postulates are given for l-groups, in terms of the operation $a\to a^*$ (the positive part of a). Perhaps the most interesting consist of the identities: $((a^{-1})^*)^{-1}a^*=a$, $((ab^{-1})^*b)^*=(a^*b^{*-1})^*b^*$, $(ab^*a^{-1})^*=ab^*a^{-1}$. These axioms are proved independent. The results partly overlap those of T. Michiura [J. Osaka Inst. Sci. Tech. Part I. **1**, 117–119 (1949); these Rev. **12**, 389]. *G. Birkhoff.*

Mallol Balmaña, R. Note. Revista Mat. Hisp.-Amer. (4) **12**, 137 (1952). (Spanish)

The author gives another set of postulates for l-groups equivalent to Michiura's [cf. the preceding review].

G. Birkhoff (Cambridge, Mass.).

Camm, Ruth. Simple free products. J. London Math. Soc. **28**, 66–76 (1953).

If ρ, σ, τ are three permutations of the non-zero integers, denote by $G(\rho,\sigma,\tau)$ the group generated by a, p, b, q subject to the relations $a^ib^{\rho(i)}=b^{\tau(i)}q^{\sigma\tau(i)}$, $i=\pm1, \pm2, \cdots$. The author specifies explicitly permutations ρ, σ, τ such that $G(\rho,\sigma,\tau)$ is simple. The specification embodies arbitrary permutations on the negative integers, so that it really gives a continuum of such groups. Since a set of groups $G(\rho,\sigma,\tau)$ which are pairwise isomorphic is at most countable, we have a continuum of non-isomorphic simple groups $G(\rho,\sigma,\tau)$. Each of these is a free product, with an amalgamated subgroup, of two free groups, and so is, in particular, locally infinite. Moreover, each can be generated by two elements.

G. Higman (Manchester).

Holyoke, T. C. On the structure of multiply transitive permutation groups. Amer. J. Math. **74**, 787–796 (1952).

A transitive permutation group G on letters 1, 2, 3, $\cdots$ contains a subgroup F that fixes 1 and is transitive on the remaining letters. The author gives a necessary and sufficient condition for an extension group H, transitive on letters 0, 1, 2, 3, $\cdots$ in which G is the subgroup leaving 0 fixed. Such a group H is known to contain an element a, such that $a^2\in F$ and a transposes 0 and 1 and induces an automorphism α in F. The author's condition involves six relations involving elements of F and G, the element $a^2\in F$, the automorphism α, and two mappings β and γ of $G-F$ into itself. The group H is then constructed by means of equivalence classes of pairs of elements $[g_1, g_2]$ and $[g_3, g_4]$ of G defined so that g_1ag_2 will equal g_3ag_4 in H. The equivalence of two transitive extensions is then considered, and certain types of groups are shown to have no transitive extensions. It is conjectured that only a finite number of triply transitive groups have transitive extensions.

J. S. Frame (East Lansing, Mich.).

Mattioli, Ennio. Sui gruppi abeliani finiti. Ann. Scuola Norm. Super. Pisa (3) **6**, 51–57 (1952).

If G is a direct product of N abelian groups of order P, each of which has a set of $P-1$ automorphisms transitive over the non-neutral elements, and if $N=(P^k-1)/(P-1)$, where k is an integer greater than 2; then G has a subgroup Γ of order P^{N-k} which gives a coset decomposition of the form $G=\Gamma+\sum\Gamma R$ where R runs through the non-neutral elements of the N abelian groups. *H. A. Thurston.*

Shoda, Kenjiro. Ein Satz über die Abelschen Gruppen mit Operatoren. Proc. Japan Acad. **28**, 241–242 (1952).

An element α is said to be algebraic over the Abelian group $\mathfrak{A}$ if the combined group $\{\mathfrak{A}, \alpha\}$ contains no subgroup $\mathfrak{B}$ with $\mathfrak{A}\cap\mathfrak{B}=0$. An algebraic extension of $\mathfrak{A}$ is generated by algebraic elements. Two extensions are equivalent when they are isomorphic by a correspondence leaving the elements in $\mathfrak{A}$ invariant. The author gives a criterion when an extension cannot be equivalent to a proper subgroup.

O. Ore (New Haven, Conn.).

Kertész, A. On the decomposability of abelian p-groups into the direct sum of cyclic groups. Acta Math. Acad. Sci. Hungar. 3, 121–126 (1952). (Russian summary)

The author establishes a new criterion for a p-primary Abelian group to be the direct sum of cyclic groups. Call the elements a_i of the group G independent if $\sum_{i=1}^{n} r_i a_i = 0$ implies $r_i a_i = 0$; and call a set S of elements of G independent if every finite subset of S is independent. Then Zorn's lemma ensures the existence of a maximal independent system S in G. Call such a system principal if no element in it can be replaced by an element of greater height without destroying the independence. (In a principal system, if it exists, the elements of finite height must obviously all be of order p.) The author's criterion is then simply: An Abelian p-group G without elements of infinite height is the direct sum of cyclic groups if and only if G possesses a principal system. From this theorem there follow as easy corollaries: (i) the basis theorem for finite groups; (ii) an Abelian p-group in which the orders are bounded is the direct sum of cyclic groups; (iii) an Abelian p-group is the direct sum of cyclic groups if and only if it is the union of a countable ascending chain of subgroups with bounded heights [Kulikov's criterion: Mat. Sbornik N.S. **16**(**58**), 129–162 (1945); these Rev. **8**, 252]; (iv) a countable Abelian p-group is the direct sum of cyclic groups if and only if it has no elements of infinite height [Prüfer's theorem: Math. Z. 17, 35–61 (1923)]. *K. A. Hirsch* (London).

Kaloujnine, L. Über eine Verallgemeinerung der p-Sylowgruppen symmetrischer Gruppen. Acta Math. Acad. Sci. Hungar. 2, 197–221 (1951). (Russian. German summary)

This is the full version of the author's note in C. R. Acad. Sci. Paris **224**, 1097–1099 (1947) [these Rev. **8**, 436]. The proofs are given completely. The presentation of the topological part of the theory is improved. The survey over the characteristic subgroups of the p-Sylow groups of the symmetric groups $\mathfrak{S}_{p^n}$ [see Kaloujnine, Ann. Sci. Ecole Norm. Sup. (3) **65**, 239–376 (1948); these Rev. **10**, 505] can be carried over to the group $\mathfrak{P}_\infty$ investigated by the author if one understands by the term "characteristic" a closed subgroup of $\mathfrak{P}_\infty$ which is mapped on itself by all continuous automorphisms of $\mathfrak{P}_\infty$. *K. A. Hirsch* (London).

Černikov, S. N. On the theory of infinite p-groups. Doklady Akad. Nauk SSSR (N.S.) **50**, 71–74 (1945). (Russian)

A p-group is called complete provided, for each natural n, every element is a product of p^nth powers of elements of the group. Every complete p-group with an ascending central series is abelian. In the infinite p-group $\mathfrak{P}$ suppose that, for each order, the set of elements having the specified order is finite; then the center of $\mathfrak{P}$ (1) has finite index, (2) contains a subgroup of finite index which is a direct product of finitely many primary abelian groups of type p^∞, and (3) contains a finite subgroup $\mathfrak{N}$ such that $\mathfrak{P}/\mathfrak{N}$ is a direct product of finitely many primary abelian groups of type p^∞ and a finite p-group. *R. A. Good* (College Park, Md.).

Gluškov, V. M. On the central series of infinite groups. Mat. Sbornik N.S. **31**(**73**), 491–496 (1952). (Russian)

In a group with an ascending [descending] central series, a so-called ZA-group [ZD-group], the length of the upper [lower] central series is called the ZA-class [ZD-class] of the group. Several questions generalizing to the transfinite case known facts about groups with finite ZA-class or ZD-class are answered. Foremost is the result that for every ordinal γ, there exists a ZA-group with ZA-class γ. But, moreover, there exists such a group with the additional property of being a ZD-group with ZD-class not exceeding ω. This contrasts the fact that a ZA-group which has a finite ZD-class must have its ZA-class also finite. For every ordinal $\gamma \geqq \omega+1$, there exists a ZA-group with ZA-class γ which is not a ZD-group. Since [according to the (unavailable) dissertation of D. M. Smirnov, Ivanovskiĭ Gos. Pedagog. Inst., 1951] the ZD-class of a ZA-group with ZA-class ω can not exceed $\omega+1$, only two possibilities exist for the ZD-class, namely, ω and $\omega+1$; examples show that each possibility can be realized. If one of the maximal abelian normal subgroups $\mathfrak{A}$ of a ZA-group $\mathfrak{G}$ is contained in a hypercenter with finite ordinal, then the ZA-class of $\mathfrak{G}$ is known [using the cited dissertation] to be finite; by contrast, if $\mathfrak{A}$ is contained in a hypercenter with ordinal not exceeding ω, then the ZA-class of $\mathfrak{G}$ can be arbitrary. All these results follow readily from the main theorem asserting, for arbitrary ordinal γ and arbitrary field P, the existence of a group of type P_γ. A complete description of the so-called type P_γ is too lengthy to be cited here. *R. A. Good.*

Lazard, Michel. Sur certaines suites d'éléments dans les groupes libres et leurs extensions. C. R. Acad. Sci. Paris **236**, 36–38 (1953).

Let G be the free group generated by e_k $(1 \leqq k \leqq r)$, $\bar{G}$ the completion of G with respect to the topology determined by the lower central series $G = G_1 \supset G_2 \supset \cdots$ of G, $x_{i,j}$ $(1 \leqq j \leqq d_i)$ elements of G_i corresponding to a basis of the free Abelian group G_i/G_{i+1}. Let L be the free Lie algebra over the rational number field generated by the e_k, $\bar{L}$ the usual completion of L (algebra of formal series), $\omega(x)$ the degree of the first non-vanishing homogeneous component of $x \in \bar{L}$. $\bar{L}$ becomes a group with the multiplication

$$xy = x + y + \tfrac{1}{2}[x, y] + \cdots$$

determined by the Campbell-Hausdorff formula and G may be regarded as the subgroup generated by the e_k. Let M be the set of all elements of $\bar{L}$ of the form $\sum_{i,j} \lambda_{i,j} x_{i,j}$ where $\lambda_{i,j}$ is a rational number involving in its denominator no prime exceeding i. A sequence $g(t)$ $(t = 0, 1, 2, \cdots)$ of elements of $\bar{G}$ (or M) is called a suite typique (typical sequence) in $\bar{G}$ (or M, respectively) if $g(t) = \sum_{k=1}^{\infty} t^k a_k$, where $a_k \in \bar{L}$ and $\omega(a_k) \geqq k$. The following results are announced. (1) $g(t) = \prod_{i,j} x_{i,j}^{H_{i,j}(t)}$ is a typical sequence in $\bar{G}$ if and only if, for all $1 \leqq i < \infty$, $1 \leqq j \leqq d_i$, $H_{i,j}(t)$ is an integer-valued polynomial in t of degree $\leqq i$. (2) The relation

$$g(t) = \prod_{k=0}^{\infty} g_k^{\binom{t}{k}}$$

defines a one-one correspondence between the set of all typical sequences in $\bar{G}$ (or M) and the set of all sequences $g_0, g_1, \cdots, g_k, \cdots$ of elements of $\bar{G}$ (or M, respectively) such that $\omega(g_k) \geqq k$, for $k \geqq 1$. The g_k are related to the a_k by simple formulae $g_1 = a_1$, $g_2 = a_2$, $g_3 = a_3 - \frac{1}{2}[a_1, a_2]$, $\cdots$. (3) M is a subgroup of $\bar{L}$ containing $\bar{G}$; and an element of $\bar{L}$ belongs to M if and only if it can be expressed as $\prod_{i,j} x_{i,j}^{\mu_{i,j}}$, where the $\mu_{i,j}$ are rational numbers involving in their denominators no prime exceeding i. Taking as typical sequence $x^t y^t$ where x and y are generators of G, (1) and (2) give the formula of P. Hall [Proc. London Math. Soc. (2) **36**, 29–95 (1933), p. 63] in a canonical form. (2) and (3) give the inversion of the Campbell-Hausdorff formula. The author also announces that he has obtained the one-one corre-

spondence between p-groups of class $<p$ and p-algebras of Lie (over the ring of integers) of class $<p$, as well as a purely algebraic proof of the results of Malcev on nilpotent groups without torsion [Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 201–212 (1949); these Rev. **10**, 507]. *P. Hall.*

Kuroš, A. G., and Černikov, S. N. Solvable and nilpotent groups. Amer. Math. Soc. Translation no. **80**, 57 pp. (1953).

Translated from Uspehi Matem. Nauk (N.S.) **2**, no. 3(19), 18–59 (1947); these Rev. **10**, 677. The translator has added footnotes, an appendix and additional bibliography, bringing up to date the status of many problems and conjectures mentioned in the paper.

Shaw, R. H. Remark on a theorem of Frobenius. Proc. Amer. Math. Soc. **3**, 970–972 (1952).

A finite group G is called a Frobenius group with respect to a subgroup H if H is its own normalizer in G and if the intersection of H with any of its conjugates in G is the identity element. Frobenius has proved, using group characters, that the subset N consisting of the identity element and the elements not in H or any of its conjugates is a normal subgroup of G. In the present paper a proof not using group characters is given for the case when H is solvable. The proof is based on the theory of transfers.

R. M. Thrall (Ann Arbor, Mich.).

Greco, Donato. Sugli omomorfismi del reticolo dei sottogruppi normali di alcuni gruppi finiti. Ricerche Mat. **1**, 185–199 (1952).

Let L be a finite-dimensional modular lattice in which if a and b cover $a\cap b$, then there exists $c\leqq a\cup b$ with $a\cap c=b\cap c=a\cap b$. Theorem: L has a proper congruence relation if and only if it has an element $(\neq 0, I)$ which is the only one of its dimension in L. The nature of such congruence relations is determined. Theorem: The lattice of normal subgroups of a finite p-group G of order p^m has a proper homomorphism if and only if there is an integer k $(0<k<m)$ such that G contains exactly one normal subgroup of order p^k. Direct products are handled in the natural way. *P. M. Whitman* (Silver Spring, Md.).

Permutti, Rodolfo. Sulle catene ad indici primi di taluni gruppi semplici. Ricerche Mat. **1**, 241–248 (1952).

Seeking to extend the results of Parker [Proc. Amer. Math. Soc. **2**, 901 (1951); these Rev. **13**, 529] on G. Birkhoff's problem 39 [Lattice theory, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 25, New York, 1948; these Rev. **10**, 673], the author obtains partial results on which groups satisfy the "Birkhoff condition" of having a chain of subgroups of prime index. In particular, almost complete results are obtained for simple groups of order not exceeding 6072. *P. M. Whitman* (Silver Spring, Md.).

Zitarosa, Antonio. Sugli elementi neutri del reticolo dei sottogruppi normali di un gruppo speciale finito. Ricerche Mat. **1**, 249–254 (1952).

An element a of a lattice L is neutral if for all $x, y \in L$ the sublattice generated by x, y, a is distributive. The author notes that if L is modular, then $a \in L$ is neutral if and only if $x, y \in L$ with $x\cup a=y\cup a$ and $x\cap a=y\cap a$ implies $x=y$. Suzuki [Trans. Amer. Math. Soc. **70**, 372–386 (1951); these Rev. **12**, 587] and Zappa [Rend. Accad. Sci. Fis. Mat. Napoli (4) **18**, 22–28 (1952); these Rev. **14**, 446] have determined the neutral elements of the lattice of all subgroups of a finite group G. The author attacks the same problem for the lattice R_n of normal subgroups. If G is the direct product of its Sylow subgroups, then a normal subgroup A of G is neutral in R_n if and only if A is the unique normal subgroup of its order. *P. M. Whitman.*

Sato, Shoji. On the lattice homomorphisms of infinite groups. I. Osaka Math. J. **4**, 229–234 (1952).

The principal results of the present paper may be summarized in the statement that the following conditions (a), (b), and (c) are equivalent: (a)

$$(S\cap T)\cup H=(S\cup H)\cap(T\cup H)$$

for every pair of subgroups S, T of G; (b) H is a normal subgroup of G, and the natural homomorphism of G onto G/H preserves the intersection of each pair of subgroups of G, and hence induces a homomorphism of the lattice of subgroups of G onto that of G/H; (c) H is normal, and if G contains elements of finite order not in H, then every element of G has finite order, and (c_1) $(\bigcap_\alpha S_\alpha)\cup H=\bigcap_\alpha(S_\alpha\cup H)$ for every [finite or infinite] set of subgroups S_α of G, whereas if G contains elements of infinite order, then (c_2) the elements of G/H have finite order, every element of finite order in G is contained in H and has order prime to the order of every element in G/H, if m, n are elements in H of infinite order then $\{m\}\cap\{n\}\neq 1$, and, for every X in G/H, x in X and n in H, there exists an element in $\{x\}\cap X$ which commutes with n. Condition (c_1) is equivalent to the property (Z) studied by the reviewer [Proc. Amer. Math. Soc. **2**, 467–478 (1951); these Rev. **12**, 800; in the reviewer's note the term "lattice homomorphism" is used to mean that every infinite, as well as finite, intersection and union is preserved]. Let M be the set of elements of finite order in G, and assume that $M\neq G$. Then G has the equivalent properties (a), (b), and (c_2) for a proper abelian normal subgroup H if and only if G/H is without elements of infinite order, M is contained in H and the center of G, the orders of the elements of M and G/H are relatively prime, and H/M is a torsion free locally cyclic group.

D. G. Higman (Montreal, Que.).

Haken, Hermann. Zum Identitätsproblem bei Gruppen. Math. Z. **56**, 335–362 (1952).

In this study of the word problem, the chief tool is the normal form of an element in the free product of groups with an amalgamated subgroup. With sufficient information on the factors and amalgamated subgroup, the word problem is solvable. Various conditions are given under which it can be decided whether a given word can be expressed without using a specified set of generators. As the author remarks, his methods enable us to solve the word problem in many cases in which Tartakovsky's [Mat. Sbornik N.S. **25(67)**, 3–50, 251–274 (1949); Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 483–494 (1949); these Rev. **11**, 493] methods do not apply. *Marshall Hall* (Columbus, Ohio).

Novikov, P. S. On algorithmic unsolvability of the problem of identity. Doklady Akad. Nauk SSSR (N.S.) **85**, 709–712 (1952). (Russian)

The author announces the theorem that there exist groups for which the word problem cannot be solved algorithmically. The latest available information is that the word problem is known to be insoluble in semi-groups [see, e.g., Post, J. Symbolic Logic **12**, 1–11 (1947); these Rev. **8**, 558] and even in semi-groups with cancellation [Turing, Ann. of Math. (2) **52**, 491–505 (1950); these Rev. **12**, 239]. On the

other hand, it is known to be soluble (beyond Magnus' well-known case of groups with one defining relation) in rather wide classes of groups [Tartakovskiĭ, Mat. Sbornik N.S. 25(67), 3–50, 251–274 (1949); Izvestiya Akad. Nauk SSSR. Ser. Mat. 13, 483–494 (1949); these Rev. 11, 493; and Haken, see the paper reviewed above].

From a known case of insolubility of Post "products" [Bull. Amer. Math. Soc. 50, 284–316 (1944); these Rev. 6, 29] the author goes over to a special kind of system ("circular system") with a finite number of generators, but an infinite number of relations for which the word problem remains insoluble. Then he shows that each circular system gives rise to a group with a finite number of generators and relations for which the word problem is equivalent to that in the circular system. The manner in which this group is constructed (by means of several alphabets and a large number of relations) is only sketched summarily, and the understanding is impaired by a number of disturbing misprints (especially in formulae (24), (24'), and (c), where an undefined symbol ρ occurs which the reviewer has been unable to identify). Since no proofs of the theorems are given, the mathematical world will have to wait in suspense until the full version of the paper is available.

K. A. Hirsch (London).

Murnaghan, F. D. On the Poincaré polynomials of the classical groups—Addendum. Proc. Nat. Acad. Sci. U. S. A. 39, 48 (1953).

This is an addendum to a previous paper by the author [same Proc. 38, 608–611 (1952); these Rev. 14, 134]. He points out that the Poincaré polynomial for the n-dimensional linear group is the generating function for self-conjugate partitions into not more than n parts. This leads directly to the result $P_n(s)=(1+s)(1+s^3)\cdots(1+s^{2n-1})$.

D. E. Littlewood (Bangor).

Hochschild, G., and Serre, J.-P. Cohomology of group extensions. Trans. Amer. Math. Soc. 74, 110–134 (1953).

Soient G un groupe, K un sous-groupe invariant de G. La recherche de relations entre les groupes de cohomologie des 3 groupes G, K, G/K, commencée par Lyndon [Duke Math. J. 15, 271–292 (1948); ces Rev. 10, 10], a été poursuivie par J. P. Serre [C. R. Acad. Sci. Paris 231, 643–646 (1950); ces Rev. 12, 272] dont les résultats sont développés et complétés ici. Théorème principal: A étant un G-module (i.e., groupe abélien dans lequel G opère), il existe une suite spectrale dont le terme E_2 est $\sum_{p,q}H^p(G/K, H^q(K, A))$ et dont le terme finale E_∞ est le groupe bigradué associé au groupe gradué $\sum_n H^n(G, A)$ convenablement filtré. Les auteurs donnent deux méthodes pour obtenir une telle suite spectrale, mais ignorent si les 2 suites spectrales obtenues sont isomorphes (une situation analogue existe dans la théorie des espaces fibrés); leurs termes E_2, en tout cas, sont isomorphes. Première méthode: soit $C(G)$ le complexe homogène du groupe G; le complexe des cochaînes de G, $\mathrm{Hom}^G(C(G), A)$, est identifié au sous-complexe de $\mathrm{Hom}^K(C(G), A)$ (dans lequel on fait opérer G/K) formé des éléments invariants par G/K. On est ramené à la situation suivante: étant donné un complexe M dans lequel opère un groupe Γ, on cherche la cohomologie du sous-complexe M^Γ des éléments invariants par Γ; ceci conduit à la suite spectrale de Cartan-Leray, dont la théorie est exposée ici en détail. Deuxième méthode: on définit une filtration (et même deux) dans le complexe des cochaînes non homogènes de G; les calculs sont plus longs, mais permettent d'expliciter certains homomorphismes de la suite spectrale.

Le chapitre III est consacré à 3 applications: 1) un théorème de décomposition directe de $H^n(G, A)$ (compatible avec les structures multiplicatives) lorsque G est un groupe fini, les ordres de G/K et K étant premiers entre eux; 2) une application à la théorie des algèbres simples, avec interprétation de l'obstruction de Teichmüller à l'aide d'un homomorphisme de "transgression", et compléments aux résultats connus; 3) le "cup product reduction theorem" est obtenu comme cas particulier d'un résultat plus général, déduit de l'homomorphisme de la suite spectrale:

$$H^{n-1}(G/K, H^1(K, A))\to H^{n+1}(G/K, A^K).$$

H. Cartan (Paris).

Borevič, Z. I. On homology groups connected with a free group. Izvestiya Akad. Nauk SSSR. Ser. Mat. 16, 365–384 (1952). (Russian)

Let g be an isomorphism of a group G onto a factor group F/R of a free group F; for each $\sigma \in G$ let u_σ be an element of F such that $g(\sigma)=u_\sigma R$; and let R_0 be the factor group of R by its commutator group (so that R_0 is abelian). If we define $\tilde{r}^\sigma=u_\sigma^{-1}ru_\sigma$ ($\sigma \in G$, $r \in R_0$) then G operates on R_0, in a way which is independent of the choice of the family $(u_\sigma)_{\sigma\in G}$. Let G also operate on an abelian group K. Theorem A: if G is finite and $n\geqq 3$, then $H^n(G, R_0)\approx H^{n-2}(G, J)$, where J is an infinite cyclic group on which G operates trivially. This is not in general true if G is not finite. If G is finite, then $H^1(G, R_0)=\{0\}$ and $H^2(G, R_0)$ is a cyclic group of order ord G. Theorem B [Eilenberg and MacLane, Ann. of Math. (2) 48, 51–78 (1947); these Rev. 8, 367]: if $n\geqq 3$, then $H^n(G, K)\approx H^{n-2}(G, \mathrm{Hom}(R_0, K))$. The Eilenberg-MacLane characterization of $H^2(G, K)$ is also re-established.

E. R. Kolchin (New York, N. Y.).

Borevič, Z. I. Homology groups of p-extensions of a regular local field. Izvestiya Akad. Nauk SSSR. Ser. Mat. 16, 427–436 (1952). (Russian)

Let k be an extension of finite degree of the field of p-adic numbers which does not contain a primitive pth root of unity; let K be a normal extension of k for which the Galois group G is a p-group. G operates on the multiplicative group K^* of non-zero elements of K. The author proves that if $n\geqq 3$, then $H^n(G, K^*)\approx H^{n-2}(G, J)$, where J is an infinite cyclic group on which G operates trivially. This result is part of a result obtained by J. Tate [Ann. of Math. (2) 56, 294–297 (1952); these Rev. 14, 252]. *E. R. Kolchin.*

Berman, S. D. On the theory of representations of finite groups. Doklady Akad. Nauk SSSR (N.S.) 86, 885–888 (1952). (Russian)

The author determines the number s of irreducible representations of a finite group G in an arbitrary field K whose characteristic does not divide the order h of G or the degrees n_i of the absolutely irreducible representations of G in some field $\bar{K}$, algebraically closed over K. Denote by $R(G, \bar{K})$ the group-ring of G over K, then $R(G, \bar{K})=\bar{I}_1\oplus\bar{I}_2\oplus\cdots\oplus\bar{I}_t$, where $\bar{I}_i$ is a total matrix algebra of degree n_i over $\bar{K}$, with a unit $\bar{e}_i$, a central idempotent. If ϵ is a primitive nth root of unity, where n is the least common multiple of the orders of the elements of G, then $\bar{e}_i \in R(G, K(\epsilon))$ and one has the well-known formula $\bar{e}_i=(n_i/h)\sum_{m=1}^{s}\chi_i(g_m^{-1})k_m$; here the g's are the representatives of the different classes of conjugate elements in G, n_i is the degree of $\bar{I}_i$, h_i is the number of elements in the class C_i and k_m is the sum of the elements of C_m. The author says that $\bar{e}_j$ is K-conjugate with $\bar{e}_i$ if for some

automorphism φ of $K(\epsilon)$ over K one has

$$\bar{e}_j = \varphi(\bar{e}_i) = (n_i/h) \sum_{m=1}^{t} \varphi[\chi_i(g_m^{-1})] k_m.$$

The relation of K-conjugation is a class relation by which the set $E = \{\bar{e}_1, \cdots, \bar{e}_t\}$ is split into classes $E_1, \cdots, E_p$ of K-conjugate idempotents. The system of characters defined by the idempotents belonging to a certain class E_i is called a "K-block" of characters. The author shows that the number s of the irreducible representations of G in K is equal to the number p of K-blocks of characters. This is a generalization of Schur's result [S.-B. Preuss. Akad. Wiss. **1906**, 164–184] who proved an equivalent theorem for number fields. Another characterization of the number s is obtained in the following manner: For $a, b \in G$ the author says that b is K-conjugate to a if b is conjugate to some power a^ν, $(\nu, h) = 1$, where $\varphi(\epsilon) = \epsilon^\nu$ for some authomorphism φ of $K(\epsilon)$ over K. This is again a class relation which splits G into classes $T_1, \cdots, T_q$ of K-conjugate elements, and it is shown that $q = s$. *J. Levitzki* (Jerusalem).

Monna, A. F. Sur un théorème de M. J. F. Koksma concernant la théorie des approximations diophantiques. II. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 342–355 (1951).

In an earlier paper [same Proc. **51**, 457–469 (1948); these Rev. **10**, 282] the author has shown that a theorem of Koksma on Diophantine approximation generalizes to topological spaces, the proof being only trivially modified. Here the proof is shown to apply to a yet more abstract situation to give a rather complicated enunciation. The author discusses how far the process of generalisation may be carried. *J. W. Cassels* (Cambridge, England).

Dynkin, E. B. A connection between homologies of a compact Lie group and its subgroups. Doklady Akad. Nauk SSSR (N.S.) **87**, 333–336 (1952). (Russian)

The author studies the homomorphism F of the Betti group of the compact Lie group G into that of the Lie group G^* that is induced by a homomorphism f of G into G^*. In fairly general cases, f is uniquely determined, within an automorphism of G^*, by F; specifically, this is the case if G is semi-simple, G^* is a classical group, and $f(G)$ is irreducible (as a set of linear transformations). For the case when G and G^* are classical formulas are given for F in terms of the weights of f. The work can be regarded as a study of the homologies of subgroups of a compact Lie group and from this viewpoint extends essential aspects of results of Samelson [Ann. of Math. (2) **42**, 1091–1137 (1941); these Rev. **3**, 143]. *I. E. Segal* (Chicago, Ill.).

Godement, Roger. A theory of spherical functions. I. Trans. Amer. Math. Soc. **73**, 496–556 (1952).

Spherical functions associated with the compact semi-simple Lie groups were introduced and studied with success in the now classical works of E. Cartan and H. Weyl. First examples of their analogues for such non-compact groups as the 3- and 4-dimensional Lorentz group occurred in the papers of V. Bargmann [Ann. of Math. (2) **48**, 568–640 (1947); these Rev. **9**, 133] and of I. Gelfand and M. Naimark [Izvestiya Akad. Nauk SSSR. Ser. Mat. **11**, 411–504 (1947); these Rev. **9**, 495]. In a general set-up they were first introduced by I. Gelfand [Doklady Akad. Nauk SSSR (N.S.) **70**, 5–8 (1950); these Rev. **11**, 498] and can be described roughly as functions on a (not necessarily compact) group which have simple transformation properties with respect to a given (usually compact) subgroup. These functions have since then been studied by Harish-Chandra [Proc. Nat. Acad. Sci. U. S. A. **37**, 170–173, 362–365, 366–369 (1951); these Rev. **13**, 106, 107] and by the reviewer [ibid. **37**, 529–533 (1951); these Rev. **13**, 434]. The present author makes an attempt to derive some properties of these spherical functions without using special computations depending on special properties of each of the classical groups and without using the Lie algebra. Using special computations, Gelfand and Naimark have obtained for the complex unimodular group explicit formulas for the spherical functions. On the other hand Harish-Chandra [loc. cit.] has obtained deep properties of these functions using Lie algebra methods extensively. Whether these properties can be obtained by general and purely integral methods in the large is unknown. The present paper is an attempt in this direction, but only a few of the results of Harish-Chandra are obtained and some of them in a significantly weaker form.

The paper contains also a section concerning spherical functions on groups of motion which are shown to be much easier than those on semi-simple Lie groups. It should also be mentioned that a proof is given that the spherical functions are real analytic. The proof is an adaptation of the classical Bernstein argument on elliptic differential operators combined with Schur's lemma and was already indicated by Bargmann, Gelfand and Harish-Chandra [loc. cit.]. The paper concludes with an appendix on the finite-dimensional representations of the classical groups where it is shown that the methods of induced representations as used by Gelfand and Naimark and Mackey for the infinite-dimensional representations can also be used in the very much easier finite-dimensional case; however, this leads only to a slight modification of the methods of E. Cartan and H. Weyl and the author's claim of a simplification is hardly justified. The reviewer agrees with the author that it is a very worthwhile problem to study spherical functions by quite general and purely integral methods, but whether the deeper properties can be so obtained remains an open question.

F. I. Mautner (Baltimore, Md.).

NUMBER THEORY

***Borel, Emile. Les nombres premiers.** Presses Universitaires de France, Paris, 1953. 135 pp.

As the author points out in the preface, for the most part nothing is assumed but elementary algebra. The contents of the book are well described by the following headings. Introduction: Divisibility of integers. Chapter 1. Definitions and elementary properties of prime numbers. 2. Tables of prime numbers. 3. Congruences and quadratic residues. (The law of quadratic reciprocity is stated but not proved.) 4. Theorems of Fermat and Wilson. 5. Sums of squares. 6. Imaginary integers. Note 1. Integral divisors of polynomials. Note 2. Distribution of primes. *L. Carlitz.*

***Kraitchik, Maurice. Mathematical recreations.** 2d ed. Dover Publications, Inc., New York, N. Y., 1953. 330 pp. Clothbound $3.00; paperbound $1.60.

Reprinted from the first edition [Norton, New York, 1942].

Bang, Thøger. A function representing all prime numbers. Norsk Mat. Tidsskr. **34**, 117–118 (1952). (Norwegian)

The real number $a=\sum p_n/10^{n^2}$, where p_n is the nth prime, has the property that

$$p_n=[f(2n-1)(f((n-1)^2)a-[f((n-1)^2)a])],$$

with $f(n)=10^n$ and $[x]$ being the integral part of x. This observation is similar to that of Sierpinski [C. R. Acad. Sci. Paris **235**, 1078–1079 (1952); these Rev. **14**, 355], both of these papers having been inspired by work of W. H. Mills [Bull. Amer. Math. Soc. **53**, 604 (1947); these Rev. **8**, 567]. *I. Niven* (Eugene, Ore.).

Thompson, John. A method for finding primes. Amer. Math. Monthly **60**, 175 (1953).

***Selmer, Ernst S. Homogeneous Diophantine equations.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 296–300. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

The full details of this paper were published in Acta Math. **85**, 203–362 (1951); these Rev. **13**, 13.

Aigner, Alexander. Ein zweiter Fall der Unmöglichkeit von $x^3+y^3=z^3$ in quadratischen Körpern mit durch 3 teilbarer Klassenzahl. Monatsh. Math. **56**, 335–338 (1952).

Developing the ideas of his earlier paper [Monatsh. Math. **56**, 240–252 (1952); these Rev. **14**, 452] the author proves that $x^3+y^3=z^3$ is insoluble in $k(m^{1/3})$ if $m=\pm 2^\nu qP$; $\nu>0$, P contains only primes $\equiv 1$ (6) with 2 a cubic non-residue and $q\equiv -1$ (6) is a prime of which $(2^{1/3}-1)$ is a quadratic non-residue. This furnishes further examples of fields $k(m^{1/3})$ whose class-number is divisible by 3 but in which $x^3+y^3=z^3$ is insoluble. *J. W. S. Cassels.*

Carlitz, L. Some sums connected with quadratic residues. Proc. Amer. Math. Soc. **4**, 12–15 (1953).

A well-known theorem of Dirichlet asserts that if p is a prime $\equiv 3$ (mod 4), then $\sum_1^{(p-1)/2}(h|p)>0$, where $(h|p)$ denotes the Legendre symbol. In the present note the author indicates a generalization of this result. He shows in particular that for $p\equiv 3$ (mod 4), $(-1)^{k+1}\sum(h|p)B_{2k+1}(h/p)$ and $(-1)^k\sum(h|p)E_{2k}(2h/p)$ are positive for $k\geq 0$, while for $p\equiv 1$ (mod 4),

$$(-1)^{k+1}\sum(h|p)B_{2k}(h/p) \text{ and } (-1)^k\sum(h|p)E_{2k-1}(2h/p)$$

are positive for $k\geq 1$. In the last four sums the summations extend from $h=1$ to $h=(p-1)/2$, and $B_k(x)$, $E_k(x)$ denote the Bernoulli and Euler polynomials, respectively, of degree k. The method of proof employs Gauss sums and the infinite series expansions of $B_k(x)$ and $E_k(x)$. *A. L. Whiteman.*

Carlitz, L. Congruences connected with the power series expansions of the Jacobi elliptic functions. Duke Math. J. **20**, 1–12 (1953).

Let p be a prime >2 and let the rational number l be integral (mod p). Put $l=k^2$, $a_m=A_m(k^2)$ in the Jacobi elliptic function $\operatorname{sn} x=\sum_1^\infty A_m(k^2)x^m/m!$, where the $A_m(k^2)$ are polynomials in k^2 with integral coefficients. Put also $x/\operatorname{sn} x=\sum_0^\infty \beta_m x^m/m!$. In a previous paper [same J. **16**, 297–302 (1949); these Rev. **10**, 593] the author derived various congruence relations involving the coefficients a_m and β_m. Now he extends these results. A typical formula is

$$\sum_{s=0}^{r}(-1)^s\binom{r}{s}A_p^{(r-s)t/(p-1)}(u)A_{m+st}(u)\equiv 0 \pmod{(p^m, p^{m-r})},$$

where u is an indeterminate and $p^{s-1}(p-1)\mid t$. Another result states that if $p^r\mid m$, $p-1\nmid m$, then $\beta_m(u)\equiv 0$ (mod p^r), that is, every coefficient of $\beta_m(u)\equiv 0$. Next put $A_m(u)=\sum_i a_{mi}u^i$; then $a_{m+t,i}\equiv(-1)^{\frac{1}{2}t}a_{mi}$ (mod (p^m, p^{m-l})) provided $p^{s-1}(p-1)\mid t$, $p^{t-1}\leq i\leq p^t$. There is a corresponding formula for the coefficients β_{mi} in the expansion $\beta_m(u)=\sum_i\beta_{mi}u^i$. Finally the author derives analogous congruence properties for the coefficients of powers of $\operatorname{sn} x$. The method of proof of these results is an elaboration of that used in the earlier paper. Of elliptic functions the author requires only the differential equation and the general form of the power series expansions and the (real) multiplication formulas. *A. L. Whiteman* (Princeton, N. J.).

Whiteman, Albert Leon. Finite Fourier series and equations in finite fields. Trans. Amer. Math. Soc. **74**, 78–98 (1953).

The principal object of the paper is the application of the method of multiple finite Fourier series to the theory of equations in a finite field. Let $\mu_1, \cdots, \mu_s$ be a set of s non-negative integers and $m_1, \cdots, m_s$ a set of s positive integers. Let $\alpha_j=e^{2\pi i/m_j}$ be an m_jth root of unity. An arbitrary function $f(\alpha_1^{\mu_1}, \cdots, \alpha_s^{\mu_s})$ can be expanded in the form

$$f(\alpha_1^{\mu_1}, \cdots, \alpha_s^{\mu_s})=\sum_{j_1, \cdots, j_s} g(j_1, \cdots, j_s)\prod_{k=1}^{s}\alpha_k^{\mu_k j_k},$$

where $0\leq j_k<m_k$; the coefficients $g(j_1, \cdots, j_s)$ and the finite Parseval relation are readily obtained. Next let g be a generator of the multiplicative group of $GF(p^n)$ and define $\operatorname{ind} a$ by means of $g^{\operatorname{ind} a}=a$ $(a\neq 0)$. Put

$$\psi(\alpha_1^{\mu_1}, \cdots, \alpha_s^{\mu_s})=\sum_{a_1, \cdots, a_{s-1}}\alpha_s^{\mu_s \operatorname{ind} A}\prod_{j=1}^{s-1}\alpha_j^{\mu_j \operatorname{ind} a_j}$$

$$(A=1-a_1-\cdots-a_{s-1}),$$

$\psi(\alpha_1^{\mu_1})=1$ for $s=1$; this is Vandiver's generalization of the Jacobi-Cauchy cyclotomic function.

In the next place let $(j_1, \cdots, j_s)$ denote the number of solutions $\gamma_1, \cdots, \gamma_s$ of the equation

$$g^{j_1+m_1\gamma_1}+\cdots+g^{j_s+m_s\gamma_s}+1=0 \quad (m_k\mid p^n-1);$$

then we have

$$\psi(\alpha_1^{\mu_1}, \cdots, \alpha_s^{\mu_s})=\epsilon(\alpha_1^{\mu_1}, \cdots, \alpha_s^{\mu_s})\sum_{j_1, \cdots, j_s}(j_1, \cdots, j_s)\prod_{k=1}^{s}\alpha_k^{\mu_k j_k},$$

where

$$\epsilon(\alpha_1^{\mu_1}, \cdots, \alpha_s^{\mu_s})=\alpha_1^{\mu_1 \operatorname{ind}(-1)}\cdots\alpha_s^{\mu_s \operatorname{ind}(-1)}=\pm 1.$$

Also put

$$\psi(\alpha_1^{\mu_1}, \cdots, \alpha_s^{\mu_s})=\epsilon(\alpha_1^{\mu_1}, \cdots, \alpha_s^{\mu_s})\Psi(\alpha_1^{\mu_1}, \cdots, \alpha_s^{\mu_s}).$$

Then the Parseval relation previously mentioned yields

$$(*)\quad \prod_{i=1}^{s}m_i\sum_{j_1, \cdots, j_s}(j_1, \cdots, j_s)(j_1+k_1, \cdots, j_s+k_s)$$

$$=\sum_{\mu_1, \cdots, \mu_s}|\Psi(\alpha_1^{\mu_1}, \cdots, \alpha_s^{\mu_s})|\prod_{i=1}^{s}\alpha_i^{-\mu_i k_i}.$$

The right member of (*) is evaluated in Theorem 1 of the present paper. This had been done by Vandiver in the case $s=2$.

The latter part of the paper is concerned with the finite Fourier expansion of the Ψ-function in the special case $\alpha_1=\cdots=\alpha_s$; the coefficients are seen to be certain sums discussed by Dickson and Hurwitz. These sums are here evaluated in the case $\mu_1=\cdots=\mu_{s-1}=0$. Also some quadratic relations of the Parseval type are evaluated. In view of the connection of certain special Dickson-Hurwitz sums with

Jacobsthal sums, it is pointed out that Theorem 4 of the present paper may be regarded as a sweeping generalization of a theorem of the author on Jacobsthal sums [Amer. J. Math. 74, 89–99 (1952), Theorem 1; these Rev. 13, 626].

L. Carlitz (Durham, N. C.).

Lomadze, G. A. On the simultaneous representation of two whole numbers by sums of whole numbers and their squares. Akad. Nauk Gruzin. SSR. Trudy Mat. Inst. Razmadze 18, 153–181 (1951). (Russian. Georgian summary)

Let $r_s(n, m)$ denote the number of solutions of the simultaneous equations $x_1^2+\cdots+x_s^2=n$, $x_1+\cdots+x_s=m$ in integers $x_1, \cdots, x_s$. Precise formulae for $r_s(n, m)$ were obtained by H. D. Kloosterman for $s=5, 7$ [Math. Ann. 118, 319–364 (1942); these Rev. 5, 33; 9, 735] and by P. Bronkhorst for $s=6, 8$ [Thesis, Groningen, 1943]. Both these writers used the theory of modular functions to establish their results. In the paper under review the author obtains all these formulae for $r_s(n, m)$ by relying solely on elementary results in the theory of functions of a complex variable and generalizing a method due to T. Estermann which is contained in a book as yet unpublished. Moreover, all infinite series occurring in the formulae of Kloosterman and Bronkhorst are now replaced by finite expressions.

L. Mirsky (Sheffield).

Ankeny, N. C. A generalization of a theorem of Suetuna on Dirichlet series. Proc. Japan Acad. 28, 389–395 (1952).

The author proves the following converse of a result of class-field theory on the representation of the Dedekind zeta-function of a finite abelian extension of the rationals as a product of L-functions. Let $\chi_0, \chi_1, \cdots, \chi_n$ be any $n+1$ characters (mod m), not necessarily distinct, at most one of which is principal. Then, if the product of the corresponding L-functions when developed into a Dirichlet series has non-negative coefficients, it is the Dedekind zeta-function of a finite abelian extension of the rationals. This is a partial generalization of a theorem of Suetuna [Tôhoku Math. J. 27, 248–257 (1926)] who established the above result for $n=1$ and $n=2$ and who also treated the cases where more than one character is principal. It is also observed that the result holds for L-series defined in any algebraic number field.

The proof is made to depend upon the following theorem on polynomials in several variables. If

$$f(x_1, x_2, \cdots, x_t)=\sum_{j_1=0}^{m_1-1}\cdots\sum_{j_t=0}^{m_t-1} a_{j_1,\cdots,j_t}x_1^{j_1}\cdots x_t^{j_t}$$

is such that (a) $f(0, 0, \cdots, 0)=1$, (b) $a_{j_1,\cdots,j_t}$ are all non-negative rational integers, (c) for each i, the greatest common divisor of the set j_i' and m_i is 1, where j_i' runs over all j_i with at least one $a_{j_1,\cdots,j_i',\cdots,j_t}\neq 0$, (d) $f(\zeta_{m_1}^{i_1}, \zeta_{m_2}^{i_2}, \cdots, \zeta_{m_t}^{i_t})\geqq 0$ for all sets of integers $i_1, i_2, \cdots, i_t$ where $\zeta_m=e^{2\pi i/m}$, then

$$f(x_1, x_2, \cdots, x_t)=\prod_{i=1}^{t}\Big(\sum_{j=0}^{m_i-1} x_i^j\Big).$$

B. Lepson (Washington, D. C.).

Iwasaki, Koziro. Simple proof of a theorem of Ankeny on Dirichlet series. Proc. Japan Acad. 28, 555–557 (1952).

A relatively elementary proof is given of Ankeny's generalization of the theorem of Suetuna (see the preceding review). It is pointed out, however, that the proof given by Ankeny is incomplete, since his preliminary theorem on polynomials in several variables is false. This is shown by the polynomial

$$f(x_1, x_2)=1+x_1x_2+x_1^2x_2^2,$$

which satisfies the hypotheses but not the conclusion of that theorem.

The author also observes that the function

$$L(s; \chi_0)^3L(s; \chi)L(s; \bar{\chi}),$$

where χ is a primitive character of order greater than 4, is represented by a Dirichlet series with non-negative coefficients, but is not expressible as the product of Dedekind zeta-functions. Therefore it seems difficult, as Ankeny has also remarked, to obtain a general result which will include the case where the principal character appears more than once.

B. Lepson (Washington, D. C.).

Roquette, Peter. Über die Automorphismengruppe eines algebraischen Funktionenkörpers. Arch. Math. 3, 343–350 (1952).

The author corrects a flaw in the paper of K. Iwasawa and T. Tamagawa [J. Math. Soc. Japan 3, 137–147 (1951); these Rev. 13, 325], which arose from an erroneous version of the discriminant theorem for algebraic function fields. The crucial lemma concerning the existence of a non-constant element z in the function field K which is invariant for an automorphism σ which in turn does not alter a given prime divisor $\mathfrak{o}$ is proved by an analysis of the representation of σ by the module of multiples of $\mathfrak{o}^{-n}$ where n is the smallest integer which is not divisible by the smallest integer occurring as the pole order for functions which have poles at $\mathfrak{o}$ only. If the minimal polynomial of this representation of σ has distinct roots, then the order of σ is at most equal to $(g+1)(2g+1)$ (an upper estimate by means of the theorem of Riemann-Roch for the degree of K over the function field which is generated by z), whereas for multiple roots the order of σ is bounded by $p(g+1)$ where g is the genus of K and p the characteristic of K. In the latter case the existence of a multiple root implies, using the fact that, for $p=0$, K is ramified at at least 3 prime divisors over a rational field P, that $p>0$. Finally, using a corollary to the Riemann-Roch theorem, it is shown there exist for fields K of characteristic $p>0$ rational subfields P such that $p \nmid [K:\mathrm{P}]$ while p divides the ramification degree of a given prime divisor.

O. F. G. Schilling (Chicago, Ill.).

Roquette, Peter. Arithmetische Untersuchung des Abelschen Funktionenkörpers, der einem algebraischen Funktionenkörper höheren Geschlechts zugeordnet ist. Mit einem Anhang über eine neue Begründung der Korrespondenzentheorie algebraischer Funktionenkörper. Abh. Math. Sem. Univ. Hamburg 18, 144–178 (1952).

The author states in the introduction the aim that all results concerning an algebraic function field of one variable K, including certain basic theorems for the associated product of the field by itself P and the associated abelian function field A, should be derived by arithmetic methods without recourse to algebraic geometry (defined in this paper by reference to the methods which are found in the papers of van der Waerden and the recent work of Weil, for example). It is stated that, for reasons of principle, one might not trust oneself completely to the geometric approach. The author carries this program to some preliminary conclusions by furnishing ideal-theoretic proofs for the properties of certain mappings of the ordinary divisor class group of the composite KA/A on the class group of K over its field of coefficients. In this manner Hasse's program

[Jber. Deutsch. Math. Verein. **52**, 1–48 (1942); these Rev. **4**, 239] which leads to preliminary results on the algebraic structure of the abelian function field A, is pushed further. The following are treated as direct applications of the mapping theorem indicated above: the interpretation of the Galois group of A over an isomorphic subfield as a group of divisor classes in K, and the theory of divisors on P as an instrument in the theory of correspondences of K.

O. F. G. Schilling (Chicago, Ill.).

Fröhlich, A. On the class group of relatively Abelian fields. Quart. J. Math., Oxford Ser. (2) **3**, 98–106 (1952).

The author investigates the class-groups (in a wider sense) of abelian fields, using the fact that they form (written additively) a representation module of the Galois group of a field. The necessary extension of representation theory was published by the author in a previous paper [same J. (2) **1**, 270–283 (1950); these Rev. **12**, 672]. His principal results are the following.

1. Let K be an abelian extension of k, C a quotient group of the class-group of K. The order of C is supposed to be prime to the degree K/k and the unit class is invariant with respect to the Galois group of K/k. Then every invariant subgroup G of C can be written as a direct product of uniquely determined subgroups $G'(K_i)$ where K_i runs through all subfields of K cyclic over k and where no class (other than the unit) of $G'(K_i)$ can be represented by a product of ideals all belonging to proper subfields of K_i.

2. Let K be an abelian extension over k of degree l^m (l prime) and K' a subfield of K containing k. Let $C(K)$ and $C(K')$ be a class-group of K and K' respectively, modulo the ideal classes in k and the classes whose order is a power of l. Then $C(K')$ is isomorphic to the subgroup of $C(K)$, which is formed by the ideals in K'.

3. The order of $C(K)$ is a product of prime powers $\equiv 1 \pmod{l}$. In particular, the class number h of an absolutely abelian field of degree l^m has the form $h=l^m\prod_\nu p_\nu^{b_\nu}$, where each factor $p_\nu^{b_\nu}\equiv 1 \pmod{l}$.

H. Heilbronn (Bristol).

***Siegel, Carl Ludwig. Die Modulgruppe in einer einfachen involutorischen Algebra.** Festschrift zur Feier des zweihundertjährigen Bestehens der Akademie der Wissenschaften in Göttingen. I. Math.-Phys. Kl., pp. 157–167. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1951.

In the present paper the author considers previously elaborated concepts of the theory of discontinuous groups in a general algebraic setting.

Let A be a simple algebra of finite degree over the field of rational numbers which possesses an involution $X\to X^*$, and $A^{(\alpha)}$ the algebra of 2-rowed matrices with elements in A. By extension from the rational to the reals, let $A\to\bar{A}$ and $A^{(\alpha)}\to\overline{A^{(\alpha)}}$, the involution also being extended to $\bar{A}$, $\overline{A^{(\alpha)}}$. The author defines the symplectic group Σ in $\bar{A}$ as the set of all $\mathfrak{M}=\begin{pmatrix}\mathfrak{A} & \mathfrak{B}\\ \mathfrak{C} & \mathfrak{D}\end{pmatrix}$ in $\overline{A^{(\alpha)}}$ which satisfy $\mathfrak{M}^*\mathfrak{J}\mathfrak{M}=\mathfrak{J}$, where $\mathfrak{J}=\begin{pmatrix}0 & \mathfrak{E}\\ -\mathfrak{E} & 0\end{pmatrix}$, $\mathfrak{E}$ being the unit matrix, and discusses its representation by linear fractional transformations $\mathfrak{Z}\to(\mathfrak{A}\mathfrak{Z}+\mathfrak{B})(\mathfrak{C}\mathfrak{Z}+\mathfrak{D})^{-1}$.

The modular group M in O, an order in A, is then defined as the set of all elements in Σ which are units in $O^{(\alpha)}$, the corresponding order in $A^{(\alpha)}$. M is shown to be a properly discontinuous subgroup of Σ of the first kind: it possesses a fundamental region F with the following properties: (1) every compact portion of Σ is covered by a finite number of copies of F; (2) only a finite number of copies of F intersect F; (3) F has a finite volume, measured in the invariant metric. It follows that M is generated by a finite number of elements. The proof is made by utilizing the regular representation of A and $\bar{A}$ and the reduction theory of quadratic forms.

J. Lehner (Los Alamos, N. M.).

Schmetterer, Leopold. Über das Produkt zweier komplexer inhomogener Linearformen. Monatsh. Math. **56**, 339–343 (1952).

The author extends known results of Minkowski, Hlawka, Mahler, and Perron for $k(1)$, $k(i)$, $k(i\sqrt{2})$ to $k(i\sqrt{7})$ in proving the following theorem. Let $\alpha, \beta, \gamma, \delta, \xi, \eta$ denote complex numbers such that $|\alpha\delta-\beta\gamma|=1$. Then integers x, y in $k(i\sqrt{7})$ exist such that $|\alpha x+\beta y-\xi|\,|\gamma x+\delta y-\eta|\leqq 4/7$; the constant $4/7$ is the best possible. The method used is that introduced by Perron [Mat. Tidsskr. B. **1949**, 1–17; S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1945/46**, 159–165; these Rev. **10**, 593; **9**, 569].

J. F. Koksma.

van der Blij, F. Binary quadratic forms of discriminant -23. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 498–503 (1952).

Let $F_1=x^2+xy+6y^2$, $F_2=2x^2+xy+3y^2$, $F_3=2x^2-xy+3y^2$ be the reduced binary forms of discriminant -23 and $a(n, F_i)$ denote the number of representations of n by F_i. The author proves

$$a(n, F_1)=(2/3)\sum_{d\mid n}(d123)+4t(n)/3,$$

$$a(n, F_2)=a(n, F_3)=a(n, F_1)-2t(n),$$

where $t(n)$ are given by

$$\sum t(n)x^n=x\prod_1^\infty(1-x^k)(1-x^{23k}).$$

The function $t(n)$ is shown to be multiplicative and imply certain congruential properties of Ramanujan's function $\tau(n)$.

B. W. Jones (Boulder, Colo.).

Černý, Karel. On the minimum of binary biquadratic forms. Čehoslovack. Mat. Ž. **2**(**77**), 1–56 (1952). (Russian. English summary)

The author investigates the minimum of a binary quartic form $f(x, y)=a_0x^4+\cdots+a_4y^4$ for integers $x, y\neq 0, 0$, when $a_0, \cdots, a_4$ are real and the roots of the equation $f(x, 1)=0$ are all complex. His results coincide with those found by C. S. Davis [Acta Math. **84**, 263–298 (1951); these Rev. **12**, 678]. (The author remarks in a footnote that he saw the review of this paper only when his own was in course of printing.) Černý's first main result corresponds to Davis's Theorem 2′ and his second main result to Davis's Theorem 4. The parameter M of Černý is $6m$ in Davis's notation. The treatment in both papers is based on techniques developed by Mordell.

H. Davenport (London).

Tsuji, Masatsugu. Theorems in the geometry of numbers for Fuchsian groups. J. Math. Soc. Japan **4**, 189–193 (1952).

The non-euclidean metric $ds=2|dz|/(1-|z|^2)$ is introduced in the unit circle $|z|<1$. By use of an analogue of a theorem of Blichfeldt [Trans. Amer. Math. Soc. **15**, 227–235 (1914)] the following analogue of Minkowski's fundamental theorem is obtained. Let G be a Fuchsian group of transformations of $|z|<1$ into itself. Let the fundamental domain D_0 of G have finite non-euclidean area $\sigma(D_0)$. Let Δ be

the disc $|z| \leq \rho$ (where $\rho<1$) with non-euclidean area $\sigma(\Delta)=4\pi\rho^2/(1-\rho^2)$. If $\sigma(\Delta) \geq 4\sigma(D_0)+\sigma^2(D_0)/\pi$, then Δ contains a point other than $z=0$ equivalent to $z=0$ under G. Analogues of some other simple results in the geometry of numbers are given. *C. A. Rogers* (London).

Macbeath, A. M. A theorem on non-homogeneous lattices. Ann. of Math. (2) **56**, 269–293 (1952).

Let R be a closed convex set in n-dimensional space. Suppose R has an interior and that it has n tac-planes having one and only one point in common. For a lattice Λ, let $d(\Lambda)$ denote its determinant. The critical determinant $\delta(K)$ of a set K symmetrical about a point u is inf $d(\Lambda)$ for all Λ such that $\Lambda \cap K=u$. The main result is that if $\Lambda \cap R \neq 0$, there is a point x in $\Lambda \cap R$ such that $\delta(R \cap R') < d(\Lambda)$, where $R'=2x-R$ is the set of all points $2x-r$ with r in R. Further, since the volume $f(x)$ of $R \cap R'$ is at most $2^n\delta(R \cap R')$, $f(x)<2^nD(\Lambda)$. The result and its proof are a considerable generalization of some work of Čebotarev and of Chalk [Quart. J. Math., Oxford Ser. **18**, 215–227 (1947); these Rev. **9**, 413]. The proof is long, requiring most of the paper, and is mostly geometric, using results from the theory of convex sets. It involves obtaining a sequence of points of Λ for which $\delta(R \cap R')$ decreases ultimately to less than $d(\Lambda)$. Particular applications of the theorem give results of Ollerenshaw [J. London Math. Soc. **20**, 22–26 (1945); these Rev. **7**, 417] and Blaney [ibid. **23**, 153–160 (1948); these Rev. **10**, 511]. Another consequence is that every lattice Λ has a point x such that

$$0 \leq x_1-x_2^2-\cdots-x_n^2 \leq \{2^{n-2}(n+1)d(\Lambda)/A\}^{2/(n+1)},$$

where A is the volume of the unit sphere in $n-1$ dimensions. *L. Tornheim* (Ann Arbor, Mich.).

Macbeath, A. M. Non-homogeneous lattices in the plane. Quart. J. Math., Oxford Ser. (2) **3**, 268–281 (1952).

A much simpler proof of the theorem of the paper reviewed above is given for the case of the plane. The theorem is then used to prove that if $k>2$ and $d(\Lambda)=a(k-2)(k^2-4)^{\frac{1}{2}}$, then there is a point of Λ in the set $a \leq xy<(k-1)^2a$, $x \geq 0$, $y \geq 0$. That this result is best possible for integral k is proved after first showing that the critical determinant of the set $x^2-\lambda y^2+(\lambda+1)|y| \leq 1$, $|y| \leq 1$ is $\frac{1}{2}(2-\lambda)^{\frac{1}{2}}$ when $0 \leq \lambda<1$. *L. Tornheim* (Ann Arbor, Mich.).

Cassels, J. W. S. A short proof of the Minkowski-Hlawka theorem. Proc. Cambridge Philos. Soc. **49**, 165–166 (1953).

The theorem in question is that if a bounded star body K in n-dimensional space, symmetrical about the origin 0, has volume $<2\sum_{m=1}^{\infty}m^{-n}$, then there exists a lattice of determinant 1 having no point except 0 in K. The method of proof is essentially that devised by Rogers and used by Davenport and Rogers [Duke Math. J. **14**, 367–375 (1947); these Rev. **9**, 11] to prove a more precise result when K is convex. *H. Davenport* (London).

Ollerenshaw, Kathleen. On the region defined by $|xy| \leq 1$, $x^2+y^2 \leq t$. Proc. Cambridge Philos. Soc. **49**, 63–71 (1953).

Let K_t denote the region of points (x, y) for which $x^2+y^2 \leq t$, $|xy| \leq 1$, $2 \leq t$. The author determines $\Delta(K_t)$, the lower bound of the determinants of the lattices whose only point interior to K_t is the origin, for all values of t. The critical lattices, the lattices of determinant $\Delta(K_t)$ whose only point interior to K_t is the origin, are also given. Where $f_1=a_1\xi^2+2b_1\xi\eta+c_1\eta^2$, $f_2=a_2\xi^2+2b_2\xi\eta+c_2\eta^2$, $d_1=b_1^2-a_1c_1>0$, $d_2=b_2^2-a_2c_2<0$, $a_1c_2-2b_1b_2+a_2c_1=0$, the results of the paper are applied to determine the least k, in terms of d_1, d_2, so that integers ξ, η, not both zero, always exist so that max $(|f_1|, |f_2|) \leq k$. The critical form pairs, i.e., the form pairs for which max $(|f_1|, |f_2|)=k$ for at least one integer pair ξ, η, are also given. The results are complementary to the analogous results given by the author for pairs of indefinite, binary quadratic forms [same Proc. **41**, 77–96 (1945); these Rev. **7**, 51]. *D. Derry.*

Rogers, C. A. The number of lattice points in a star body. J. London Math. Soc. **26**, 307–310 (1951).

For abbreviation, by "star" we denote a set in n-dimensional space which is star-shaped with respect to the origin 0 and also symmetric with respect to 0. A "lattice" means a lattice of which 0 is a lattice point. If S is a star, then $\Delta(S)$ denotes the lower bound of the determinants $d(\Lambda)$ of the lattices Λ with no points other than 0 in S. $V(S)$ denotes the Jordan measure of S. If S is convex, then a well-known theorem of Minkowski asserts that $2^n\Delta(S) \geq V(S)$, whereas van der Corput [Acta Arith. **2**, 145–146 (1936)] proved more generally that $V(S)>m2^nd(\Lambda)$ (S convex) implies that S contains at least m distinct pairs $\pm A_1, \cdots, \pm A_m$ of points of Λ.

The author states the following conjecture (which, if true, shows among other things that van der Corput's result follows from Minkowski's). Let m be a positive integer, S a star, Λ a lattice. Then, if $\Delta(S)>md(\Lambda)$, S contains m distinct pairs $\pm A_1, \cdots, \pm A_m$ of points of Λ. He states that he is able to prove this when m is of certain special forms, and gives the proof for the case that m is a prime. In that case he even shows that either there is a point $A_1 \in \Lambda$ $(A_1 \neq 0)$ such that $pA_1 \in S$, or there are $p+1$ distinct pairs of primitive points of Λ in S.

As his conjecture is proved for a set of numbers m_1, m_2, $\cdots$ for which $m_{n+1}/m_n \to 1$, the author is able to give a simple proof for the inequality $2\zeta(n)\Delta(S) \leq V(S)$, which was stated by Minkowski and proved by Hlawka [Math. Z. **49**, 285–312 (1943); these Rev. **5**, 201]. *N. G. de Bruijn* (Amsterdam).

Rankin, R. A. The anomaly of convex bodies. Proc. Cambridge Philos. Soc. **49**, 54–58 (1953).

Let K be a convex symmetrical body in 3-dimensional space with critical determinant $\Delta(K)$. For any lattice Λ let $\mu_r=\mu_r(K, \Lambda)$ be the lower bound of the positive numbers μ such that the expanded body μK contains r linearly independent points of Λ (for $r=1, 2, 3$). It has been conjectured that $\mu_1\mu_2\mu_3\Delta(K) \leq d(\Lambda)$. The author proves that $\mu_1\mu_2\mu_3\Delta(K) \leq \beta d(\Lambda)$ where $\beta=1.226\cdots$ satisfies

$$\beta(\beta+6)^2=64.$$

The proof depends on a geometrical result of the author [Proc. Cambridge Philos. Soc. **49**, 44–53 (1953); these Rev. **14**, 678]. *C. A. Rogers* (London).

Mahler, K. On the approximation of logarithms of algebraic numbers. Philos. Trans. Roy. Soc. London. Ser. A. **245**, 371–398 (1953).

The author gives a new measure of transcendency for the logarithm of an algebraic number $\neq 0$, $\neq 1$ (Theorem 3). This important result is free of unknown numerical constants. One of its corollaries is that the logarithm of an algebraic number is not a U-number. This extends a well-known result of the author for the logarithm of a rational

number [J. Reine Angew. Math. **166**, 137–150 (1932), p. 149].

The proof of Theorem 3 uses an idea due to Siegel and is based on a system of identities of the form

$$\sum_{k=0}^{m} A_{hk}(x)(\ln x)^k = R_h(x) \quad (h=0, 1, \cdots, m),$$

where the A's are polynomials of degree not greater than n with integral coefficients and, if $x \neq 1$, with a nonvanishing determinant, while the R's have at $x=1$ a zero of order at least $(m+1)n$.

The same method enables the author to study more closely the rational approximations to the logarithms of rational numbers. In particular, he derives a very simple measure of irrationality for the logarithm of a rational number (Theorem 5). As an application he obtains a number of results of the following type: $|2^\alpha - e^{\alpha_1}| \geq |2^3 - e^3|$ for all pairs of positive integers α, α_1 and with equality only in the case $\alpha=3$, $\alpha_1=2$. Finally he derives by means of these methods inequalities, such as $\ln f - [\ln f] > f^{-40 \ln \ln f}$, $e^\alpha - [e^\alpha] > \alpha^{-40\alpha}$, where both f and α are sufficiently large positive integers. *J. Popken* (Utrecht).

ANALYSIS

Kametani, Shunzi. A new formulation of mean value theorem. Nat. Sci. Rep. Ochanomizu Univ. **1**, 1–5 (1951).

The author gives the mean-value theorem in a formulation the proof of which depends directly on the connectivity of a linear interval. He thus generalizes the theorem to the case of interval functions defined in spaces of more than one dimension, as follows: Let φ and σ be continuous additive functions of an interval on a closed interval I_0; then there exist a point x_0 and a descending sequence of intervals $\{I_j\}$ satisfying the following conditions: $I_0 \supset I_1 \supset \cdots \ni x_0$; x_0 is an interior point of each I_j; every I_j may be chosen so as to be similar to I_0; $\varphi(I_j)\sigma(I_0) = \varphi(I_0)\sigma(I_j)$ $(j=1, 2, \cdots)$. Applications are made by taking $m(I)$ for $\sigma(I)$.

E. F. Beckenbach (Los Angeles, Calif.).

Vučković, V. Quelques extensions des théorèmes de moyenne. Srpska Akad. Nauka. Zbornik Radova **18**, Matematički Inst. **2**, 159–166 (1952). (Serbo-Croatian. French summary)

Extending Karamata's result [Srpska Akad. Nauka. Zbornik Radova **7**, Matematički Inst. **1**, 119–124 (1951); these Rev. **13**, 329]

$$\frac{f(b)-f(a)}{b-a} = pf_+'(\xi) + qf_-'(\xi), \quad p>0, \ q>0, \ p+q=1, \ a<\xi<b,$$

for functions $f(x)$ which are continuous in the closed interval (a, b) and admit right and left derivatives at each point of the open interval, the author obtains the analogous result that if for functions $f(x)$ and $\varphi(x)$, continuous in the closed interval (a, b) right and left derivatives $f_+'(x)$, $\varphi_+'(x)$, $f_-'(x)$, $\varphi_-'(x)$, respectively, exist at each point of the open interval, and $\varphi_-'(x)$ and $\varphi_+'(x)$ have the same sign and do not vanish, then there exist numbers $p>0$, $q>0$, $p+q=1$, and a value ξ, $a<\xi<b$, for which

$$\frac{f(b)-f(a)}{\varphi(b)-\varphi(a)} = \frac{pf_+'(\xi)+qf_-'(\xi)}{p\varphi_+'(\xi)+q\varphi_-'(\xi)}.$$

Applications are made to finite Taylor series and to mean-value theorems for integrals. *E. F. Beckenbach.*

Barjaktarević, M. Sur le théorème de la moyenne généralisée. Bull. Soc. Math. Phys. Serbie **3**, nos. 3–4, 15–23 (1951). (Serbo-Croatian. French summary)

The author determines necessary and sufficient conditions on functions $f(t)$, $\varphi(t)$, and $\xi(x_1, x_2)$ in order that they satisfy the mean-value relation

$$\frac{f(x_2)-f(x_1)}{\varphi(x_2)-\varphi(x_1)} = \frac{f'(\xi)}{\varphi'(\xi)}.$$

In particular, he determines the class of functions $f(t)$ and $\varphi(t)$ for which $\xi(x_1, x_2)$ is of the form

$$\xi = \psi^{-1}\{\tfrac{1}{2}[\psi(x_1)+\psi(x_2)]\}.$$

Cf. R. Rothe [Math. Z. **9**, 300–325 (1921)] and R. Bojanić [Bull. Soc. Math. Phys. Serbie **1**, no. 3–4, 105–111 (1949); Acad. Serbe Sci. Publ. Inst. Math. **3**, 219–226 (1950); Glas Srpske Akad. Nauka. Od. Prirod.-Mat. Nauka **198**, 187–196 (1950); these Rev. **11**, 716; **12**, 483; **13**, 214].

E. F. Beckenbach (Los Angeles, Calif.).

Ostrowski, Alexandre. Sur quelques applications des fonctions convexes et concaves au sens de I. Schur. J. Math. Pures Appl. (9) **31**, 253–292 (1952).

The author systematically studies the transformations of Schur and obtains new proofs and improved results concerning functions which are convex or concave in the sense of Schur. A transformation of Schur is a transformation $y_\mu = \sum_{\nu=1}^n s_{\mu\nu}x_\nu$ $(\mu=1, \cdots, n)$ for which $s_{\mu\nu} \geq 0$ and $\sum_{\nu=1}^n s_{\mu\nu} = \sum_{\mu=1}^n s_{\mu\nu} = 1$ $(\mu, \nu=1, \cdots, n)$. A function $F(x_1, \cdots, x_n)$, $k>1$, is convex in the sense of Schur provided $F(y_1, \cdots, y_k) \leq F(x_1, \cdots, x_k)$ for each set (y) obtained from (x) by a transformation of Schur; an analogous definition holds for functions concave in the sense of Schur. It follows immediately that a function which is convex or concave in the sense of Schur necessarily is symmetric.

A principal tool used by the author is the simple result that, if $x_1 \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_n$, and

$$y_1+\cdots+y_k \leq x_1+\cdots+x_k \quad (k=1, \cdots, n-1, n),$$

then there exist numbers $z_1, \cdots, z_n$ for which $y_1 \leq z_1, \cdots, y_n \leq z_n$, $z_1 \geq \cdots \geq z_n$, $z_1+\cdots+z_n = x_1+\cdots+x_n$, and

$$z_1+\cdots+z_k \leq x_1+\cdots+x_k \quad (k=1, \cdots, n-1).$$

As a typical example of the results obtained in the present paper, we recall the classical theorem of Hadamard that if $\sum_{\mu,\nu=1}^n h_{\mu\nu}\bar{x}_\mu x_\nu$ is a positive definite Hermitian form, with eigenvalues $\omega_1 \leq \cdots \leq \omega_n$, then $h_{11}\cdots h_{nn} \geq \omega_1 \cdots \omega_n$; we recall also the extension by Schur, that for any function $G(x_1, \cdots, x_n)$ which is concave in the sense of Schur we have $G(h_{11}, \cdots, h_{nn}) \geq G(\omega_1, \cdots, \omega_n)$. The author shows that for each k, $1 \leq k \leq n$, if $G(x_1, \cdots, x_k)$ is concave in the sense of Schur, and increasing in $x_1, \cdots, x_k$, then

$$G(h_{11}, \cdots, h_{kk}) \geq G(\omega_1, \cdots, \omega_k).$$

Generalizations of inequalities recently obtained by H. Weyl, Ky Fan, G. Pólya, and A. Horn also are given [Proc. Nat. Acad. Sci. U. S. A. **35**, 408–411, 652–655 (1949); **36**, 31–35, 49–51, 374–375 (1950); these Rev. **11**, 37, 600, 526; **13**, 565]. *E. F. Beckenbach* (Los Angeles, Calif.).

Lorentz, G. G. An inequality for rearrangements. Amer. Math. Monthly **60**, 176–179 (1953).

Let $f_1, \cdots, f_n$ denote positive measurable functions on $(0, 1)$ and $f_1^*, \cdots, f_n^*$ their equimeasurable decreasing rearrangements. The author seeks to determine conditions on a continuous function $\Phi(x, u_1, \cdots, u_n)$ defined for $0<x<1$, $u_k \geqq 0$, $k=1, \cdots, n$, under which

$$(*) \quad \int_0^1 \Phi(x, f_1(x), \cdots, f_n(x))dx \leqq \int_0^1 \Phi(x, f_1^*(x), \cdots, f_n^*(x))dx$$

for any set of positive bounded measurable functions $f_1, \cdots, f_n$. If $\Phi \in C''$, it is found that (*) holds if and only if $\partial^2\Phi/\partial u_i \partial u_j \geqq 0$ and $\partial^2\Phi/\partial x \partial u_i \leqq 0$. If Φ is merely continuous, necessary and sufficient conditions are given in terms of second differences. For example, (*) holds for $\Phi = u_1 \cdots u_n$, and for $\Phi = F(u_1 + \cdots + u_n)$ if and only if F is convex.

W. Rudin (Rochester, N. Y.).

Polya, Georges. Sur le rôle des domaines symétriques dans le calcul de certaines grandeurs physiques. C. R. Acad. Sci. Paris **235**, 1079–1081 (1952).

Let D be a simply connected plane domain. A is its area, I its moment of inertia with respect to the centroid, $\dagger$ its maximum interior conformal radius, p its torsional rigidity if it is regarded as the cross section of a bar, C its electrostatic capacity, regarding it as a conducting plate, and Λ_i, $i=1, 2, 3, \cdots$, its successive characteristic frequencies, regarding it as a drumhead. The same quantities relative to another region D_0 are A_0, I_0, etc. If the transformation $x=ax_0$, $y=by_0$ maps D_0 into D, then D and D_0 are said to be related by a dilatation, and the coordinate axes are called the axes of the dilatation. If D is transformed into itself by a rotation of amount $2\pi/n$, D is said to have rotational symmetry of order n.

The note is devoted to the announcement and application of several inequalities between the quantities listed above, as follows. I. If D and D_0 are related by a dilatation and D_0 has an axis of symmetry equally inclined to the axes, then $A^4/IP \leqq A_0^4/I_0P_0$, $C^2A/I \leqq C_0^2A_0/I_0$, and $\Lambda_1^2A^2/I \leqq \Lambda_{1,0}^2A_0^2/I_0$. II. If D and D_0 are related by any affine transformation and D_0 has at least three axes of symmetry, the inequalities in I remain valid. III. If D_0 is a circle of radius 1, $\Lambda_1\dagger \leqq \Lambda_{1,0}$. If D is convex and has a rotational symmetry of order $\geqq 3$, then $\Lambda_2\dagger \leqq \Lambda_{2,0}$ and $\Lambda_3\dagger \leqq \Lambda_{3,0}$. If in addition D has a rotational symmetry of order $\geqq 5$, then also $\Lambda_i\dagger \leqq \Lambda_i$ for $i=4, 5, 6$. Application of those inequalities is made to a triangle and explicit bounds are found for various of the quantities discussed. *J. W. Green.*

Birindelli, Carlo. Su nuove formule interpolatorie del Picone per funzioni in più variabili e loro contributo al calcolo numerico degli integrali multipli. Compositio Math. **10**, 117–167 (1952) = Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **349** (1952).

Extension à l'espace R^3 de la méthode d'interpolation centropériphérique par la dérivée totale selon Picone qui l'a exposée dans R^2 [Ann. Scuola Norm. Super. Pisa (3) **5**, 193–244 (1951); ces Rev. **14**, 144]; application à l'extension de la formule de Cavalieri-Simpson; évaluations des restes. Il appartient aux calculateurs de dire si les formules préconisées sont pratiques, bien qu'assez compliquées.

J. Favard (Grenoble).

★Bang, Thøger. Sur les points singuliers (dans un sens généralisé) des fonctions indéfiniment dérivables. Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 259–263. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

Let $f(x)$ be of class C^∞ on the real axis R. Let $\{m_n\}$ be a positive sequence such that $\{m_n/nm_{n-1}\}$ is nondecreasing, and put $1/r(x) = \limsup \{|f^{(n)}(x)|/m_n\}^{1/n}$. Then $R = A + C + P$, where A is the set of points x such that $\liminf_{y \to x} r(y) > 0$, C is the set where $r(x) > 0$ but $\liminf_{y \to x} r(y) = 0$, and P is the set where $r(x) = 0$. When $m_n = n!$, these are respectively the sets of points (A) where $f(x)$ is analytic, (C) where the Taylor series has positive radius of convergence but fails to represent the function in a neighborhood of x, (P) where the Taylor series diverges except at x. The author sketches a proof that three disjoint and exhaustive sets A, C, P are respectively the sets where some $f(x)$ has properties A, C, P if and only if A is open, $R-C$ is everywhere dense, and P is a G_δ. The case $m_n = n!$ was considered by Zahorski [Fund. Math. **34**, 183–245 (1947); these Rev. **10**, 23], but his proof of the sufficiency of the conditions was more difficult. The same results hold if lim sup is replaced by sup in the definition of $r(x)$. *R. P. Boas, Jr.* (Evanston, Ill.).

Puig Adam, P. Some generalizations of the algorithm of continued fractions of differential elements. Revista Mat. Hisp.-Amer. (4) **12**, 206–222 (1952). (Spanish. French summary)

This paper contains a résumé of theorems given in an earlier paper [same Revista (4) **11**, 180–190 (1951); these Rev. **13**, 540], together with a more detailed proof of one of them, and extensions of the algorithm, e.g., to functions f and g regular in a domain in the complex plane.

H. S. Wall (Austin, Tex.).

MacNerney, J. S. Half-bounded matrices. J. Indian Math. Soc. (N.S.) **16** (1952), 151–176 (1953).

The numerical range $W(A)$ of a matrix A is defined as the set S such that $w \in S$ only if there exists a positive integer n and a sequence $(x_1, \cdots, x_n)$ of complex numbers such that $\sum_{p=1}^n |x_p|^2 = 1$ and $w = \sum_{p,q=1}^n x_p A_{pq} \bar{x}_q$. A half-bounded matrix is a matrix M such that $W(M)$ is not the entire plane; since the numerical range of a matrix is a convex set, a half-bounded matrix is a matrix M whose numerical range is contained in some half-plane. Thus, every bounded matrix and every Hermitian matrix is half-bounded, and the matrix associated with a positive definite J-fraction is half-bounded.

This contribution to the theory of infinite matrices is concerned with (1) the existence of a bounded reciprocal of $zI-A$, for A half-bounded and completely squarable and z at a positive distance from $W(A)$, (2) extension of the theory of Stieltjes transforms of nondecreasing functions, especially as related to J-fractions and the moment problem, to a theory of Stieltjes transforms of "nondecreasing matrices", especially as related to completely squarable and half-bounded matrices and the matrix moment problem, and (3) application of the foregoing to the settling of certain problems concerning equivalent functions of positive definite J-fractions. A brief sketch of some of the results follows.

Suppose A is a half-bounded and completely squarable matrix and $d(z)$ is the distance from z to $W(A)$. For $d(z) > 0$, there exists a matrix $B(z)$ of regular functions, which is bounded by $1/d(z)$, is symmetric if A is symmetric, and is a reciprocal of $zI-A$. A reciprocal $B(z)$ of $zI-A$, for z in some unbounded set Z, determines A in various asymptotic

senses depending on types of boundedness assumed for $B(z)$. For instance, if $B(z)$ has norm $b/|z|$, where b is a positive number, then $z[zB(z)_{pq}-I_{pq}]\to A_{pq}$ as $z\to\infty$ in Z. These ideas are extended to reciprocals of $zI-A-E(z)$, where $E(z)$ is a half-bounded matrix of regular functions.

A nondecreasing matrix is a matrix $G(t)$ of complex-valued functions over the real numbers, such that $\sum_{p,q=1}^{n}x_pG(t)_{pq}\bar{x}_q$ is a bounded and nondecreasing function $g(t)$ over the real numbers ($n=1, 2, \cdots$, x_r any complex number), and $g(t)=g(t+)-g(-\infty)$. A unit nondecreasing matrix is a nondecreasing matrix $G(t)$ such that $G(+\infty)=I$. If $F(z)$ is a matrix of complex-valued functions defined for $\operatorname{Im} z>0$, then, in order that $\sum_{p,q=1}^{n}x_pF(z)_{pq}\bar{x}_q$ ($n=1, 2, \cdots$, x_r any complex number) be a function $f(z)$, regular, $\operatorname{Im} f(z)\leqq 0$, and $zf(z)$ bounded as $\operatorname{Im} z\to+\infty$ with $\operatorname{Re} z=0$, it is necessary and sufficient that $F(z)$ be the Stieltjes transform $\int_{-\infty}^{+\infty}(z-t)^{-1}dG(t)$ of a nondecreasing matrix $G(t)$. A known sufficient condition for a function $f(z)$, regular for $\operatorname{Im} z>0$, to be the Stieltjes transform of a nondecreasing function g with $g(+\infty)-g(-\infty)=1$ is extended to the necessary and sufficient condition: $|f(z)-(z-\bar{z})^{-1}|\leqq|(z-\bar{z})^{-1}|$ for $\operatorname{Im} z>0$, and the analog for matrices is obtained. If $F(z)$ is the Stieltjes transform of the unit nondecreasing matrix $G(t)$, $A=\int_{-\infty}^{+\infty}tdG(t)$, $B=\int_{-\infty}^{+\infty}t^2dG(t)$, and B is bounded, then there exists a nondecreasing matrix $E(t)$ such that

$$E(+\infty)=B-AA$$

and, for $\operatorname{Im} z>0$, $zI-A-\int_{-\infty}^{+\infty}(z-t)^{-1}dE(t)$ is the bounded reciprocal of $F(z)$.

The problems about J-fractions concern two conjectures on p. 318 of the reviewer's book, Analytic theory of continued fractions [Van Nostrand, New York, 1948; these Rev. 10, 32]. Also, there is an addition to Theorem 84.1, p. 316. *H. S. Wall* (Austin, Tex.).

Cinquini-Cibrario, Maria. Metodi esistenziali in analisi matematica. Atti Sem. Mat. Fis. Univ. Modena **5**, 90–100 (1951)

Introductory lecture to a course in mathematical analysis.

Theory of Sets, Theory of Functions of Real Variables

***Vogel, Alfred. Klassische Grundlagen der Analysis.** S. Hirzel Verlag, Leipzig, 1952. x+194 pp. 8.50 DM.

This is a very carefully written, always well motivated, and easily readable discussion of the system of real numbers. The Peano axioms for the natural numbers form the starting point. Then the absolute-rational numbers, the rational numbers, and, by means of nested intervals, the real numbers are successively introduced. Each time the essential rôle of equivalence classes is stressed. A short last chapter discusses also transfinite cardinal numbers.

A. Rosenthal (Lafayette, Ind.).

***Sierpiński, W. Les ensembles projectifs et analytiques.** Mémor. Sci. Math., no. 112. Gauthier-Villars, Paris, 1950. 80 pp. 600 francs.

This monograph contains an elementary presentation and discussion of the underlying ideas, basic definitions, principal results, and unsolved problems concerning projective and analytic sets. It is carefully and simply written in terms requiring no special preparation on the part of the reader and is amply illustrated with easily visualized examples. Numerous alternate definitions, particularly of analytic sets, are discussed. Interesting historical remarks concerned with the origin and development of the notions involved add much to the permanent value of the monograph. These could be supplied only by this distinguished author, who has played such a key role in the subject himself. Here we have a masterpiece of organization and effective presentation of a fundamental and difficult topic which is so intimately interwoven into the foundations of mathematics.

G. T. Whyburn (Charlottesville, Va.).

Gustin, William. Partitioning an arithmetic interval. Canadian J. Math. **5**, 81–85 (1953).

Let $|(\alpha)$ denote, for any ordinal α, the set of all ordinals $<\alpha$. This paper deals with the set $|(\omega+1)$, its elements, and subsets containing 0. A proper sequence is any m-sequence d_μ ($\mu<m$, $1\leqq m\leqq\omega$) such that $1<d_\mu\leqq\omega$ and such that the ordinal product of d_μ's is >1 and $\leqq\omega$. A set X is partitioned into a sequence X_μ of sets, symbolically $X=\sum X_\mu$, provided for any $x\in X$ one has uniquely $x=\sum x_\mu$ with $x_\mu\in X_\mu$, and conversely, if the sum of each such sequence is in X. E.g., $I(d_1d_2)=I(d_1)+d_1I(d_2)$. For any proper sequence $d_1, d_2, \cdots$ (finite or infinite)

$$I(d_1d_2\cdots)=I(d_1)+d_1I(d_2)+d_1d_2I(d_3)+\cdots.$$

Any coarser partition of $I(d_1d_2\cdots)$ may be obtained by arbitrarily grouping the terms of the above partition (Theorem 1). If $n\geqq 2$ and $I(n)=\sum X_\mu$, there is a unique sequence $d_1, d_2, \cdots$ of ordinals >1 such that $n=d_1d_2\cdots$ and such that the X_μ-partition may be generated by appropriate grouping of terms of the partition $I(n)=I(d_1)+d_1I(d_2)+\cdots$ so that the consecutive terms of this partition be allocated to different X_μ-sets. The recursion formula

$$p_m(n)=1+(m-1)\sum_{d|n}p_m(d)$$

for the number $p_m(n)$ of all partitions of $I(n)$ into m-sequences of sets is established [cf. Proc. Amer. Math. Soc. **3**, 31–35 (1952); these Rev. **14**, 138]. *G. Kurepa.*

Urciuolo, Victorio. The general theory of number. Revista Univ. Nac. Córdoba **38**, no. 1, 423–481 (1951). (Spanish)

This monograph is a sketch of the number-theoretic part of a full work on general set theory, in preparation. It consists of a discussion introducing—via considerations on set and order attributed in broad outline to Frege—the natural numbers, the rational numbers, the real numbers, and the n-dimensional vectors with real components (termed "complex numbers"). The main novelty of the treatment resides in the author's unorthodox view as to the general notion of set appropriate for developing the theory of number. According to this view it is, e.g., inherent in this notion of set that a set may not contain any subsets of itself as elements, except that a one-element set is to be identified with its single element. The author promises a more thoroughgoing exposition of his ideas, which he believes have succeeded in avoiding the classical paradoxes of class membership.

T. A. Botts (Charlottesville, Va.).

Haupt, Otto. Zur Differentiation additiver Funktionen. Math. Nachr. **8**, 93–97 (1952).

The differentiation of (finitely) additive set functions was studied by S. Kempisty [Fonctions d'intervalle non additives, Hermann, Paris, 1939; these Rev. **1**, 207] and P. Romanovski [Mat. Sbornik N.S. **9(51)**, 67–120 (1941); these Rev. **2**, 354]. The author now develops a generaliza-

tion of this theory such that no topology (in the basic set) has to be assumed. A suitable system of axioms is given and some theorems (without proofs) are stated. A more detailed presentation, including the proofs, will be published in volume III of the second edition of Haupt-Aumann-Pauc, Differential- und Integralrechnung. *A. Rosenthal.*

Aumann, Georg. Zur Spiegelungsinvarianz des Lebesgueschen Masses. Arch. Math. **3**, 360 (1952).

A simple proof is given for the invariance of the Lebesgue measure with respect to reflections. *A. Rosenthal.*

Džvaršeišvili, A. G. On a sequence of integrals in the sense of Denjoy. Akad. Nauk Gruzin. SSR. Trudy Mat. Inst. Razmadze **18**, 221–236 (1951). (Russian. Georgian summary)

A sequence of functions $F_1(x), F_2(x), \cdots$ defined on the interval (a, b) is said to be uniformly (AC*) [(AC)] on the set $E \subset (a, b)$ if given $\epsilon > 0$ there is an $\eta > 0$ such that for every system of intervals $\{\delta_k\}$ whose end-points belong to E the inequality $\sum_1^m |\delta_k| < \eta$ implies

$$\sum_1^m \omega(F_n; \delta_k) < \epsilon \quad \left[\sum_1^m |\Delta(F_n; \delta_k)| < \epsilon\right] (n = 1, 2, \cdots),$$

where $\omega(F_n; \delta_k)$ [$\Delta(F_n; \delta_k)$] denotes the oscillation [the increment] of $F_n(x)$ on the interval δ_k. The sequence $\{F_n(x)\}$ is uniformly (ACG*) [(ACG)] on (a, b) if every perfect set $E \subset (a, b)$ contains a portion $P \subset E$ such that $\{F_n(x)\}$ is uniformly (AC*) [(AC)] on P.

Theorem: Let $f_1(x), f_2(x), \cdots$ be a sequence of functions which are (D)-integrable [(D^*)-integrable] on (a, b) and converge to a limit function $f_0(x)$. If the sequence $\{F_n(x) = \int_a^x f_n(t)dt\}$ is uniformly (ACG) [(ACG*)] and uniformly continuous on (a, b), then the limit function $f_0(x)$ is (D)-integrable [(D^*)-integrable] and

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f_0(x)dx.$$

The author obtains several other conditions for the term-by-term integration and gives the following application to Fourier-Denjoy series. Let $f(x)$ be (D)-integrable, $g(x)$ of bounded variation on $(0, 2\pi)$, and let a_n, b_n and a_n', b_n' be the Fourier coefficients of $f(x)$ and $g(x)$ respectively. Then the series $\frac{1}{2}a_0a_0' + \sum_1^\infty (a_na_n' + b_nb_n')$ is summable (C, α) to the value $\pi^{-1}\int_0^{2\pi} f(x)g(x)dx$, for $\alpha > 0$. *M. Cotlar.*

Džvaršeišvili, A. G. On integration and differentiation under the Denjoy integral sign. Soobščeniya Akad. Nauk Gruzin. SSR **12**, 385–392 (1951). (Russian)

Let $f(x, y)$ be a measurable function defined on the 2-dimensional interval $[a \leq x \leq b, c \leq y \leq d]$ and separately (D)-integrable in each variable. Theorem: Let $E \subset (c, d)$ be a measurable set with the following properties: (1) for every interval $(\alpha, \beta) \leq (a, b)$ the function $\phi(y) = \int_\alpha^\beta f(x, y)dx$ is summable on E and $\int_E \phi(y)dy$ is a continuous function of the interval (α, β); (2) every closed set $\phi \subset (a, b)$ contains a portion P such that $f(x, y)$ is summable for $x \in P$, $y \in E$; (3) if $\{(\alpha_k, \beta_k)\}$ are the intervals contiguous to P, then the sequence of integrals $\{\int_E dy \sum_{k=1}^n \int_{\alpha_k}^{\beta_k} f(x, y)dx\}$ is uniformly absolutely continuous on E and $\sum_{k=1}^\infty |\int_{\alpha_k}^{\beta_k} f(x, y)dx|$ converges for almost all $y \in E$. Then the integral $\int_\alpha^\beta dx \int_E f(x, y)dy$ exists and $\int_\alpha^\beta dx \int_E f(x, y)dy = \int_E dy \int_\alpha^\beta f(x, y)dx$. Several other theorems of the same kind are given. *M. Cotlar.*

Ellis, H. W. On the relation between the P^2-integral and the Cesàro-Perron scale of integrals. Trans. Roy. Soc. Canada. Sect. III. (3) **46**, 29–32 (1952).

R. D. James [Canadian J. Math. **2**, 297–306 (1950); these Rev. **12**, 94] has defined a second Perron integral (P^2-integral) which is more general than the Cesàro-Perron integral. The Cesàro-Perron integral is the integral of order one in the scale of integrals C_rP defined by Burkill [Proc. London Math. Soc. (2) **39**, 541–552 (1935)]. The author gives examples showing that there are functions that are P^2-integrable without being C_rP-integrable for any r and that, for r an integer greater than one, there are functions that are C_rP-integrable without being P^2-integrable.

M. Cotlar (Buenos Aires).

García Pradillo, Julio. Product integrals. Gaceta Mat. (1) **4**, 74–80 (1952). (Spanish).

Expository paper.

Baiada, Emilio. Un criterio di convergenza in lunghezza e la derivazione per serie. Ann. Scuola Norm. Super. Pisa (3) **6**, 59–68 (1952).

This paper is devoted to establishing the following theorem. Let $\{f_n\}$ be a sequence of real functions on the real interval (a, b), continuous together with their first derivatives (or more generally merely absolutely continuous), and satisfying the condition

$$\int_a^{b-h} |f_n'(x+h) - f_n'(x)|dx \leq e_n(h),$$

where $e_n(h)$ tends to zero with h and satisfies the condition $\sum_{r=1}^\infty e_n(h/2^r) \leq \sigma(h)$ uniformly in n, $\sigma(h)$ tending to zero with h. If the sequence $\{f_n\}$ converges uniformly on (a, b) to a function f (which is therefore continuous) of bounded variations, then the sequence $\{f_n'\}$ of derivatives converges in measure on (a, b) to f', and the length of the curve $y = f_n(x)$ converges to the length of the curve $y = f(x)$.

T. A. Botts (Charlottesville, Va.).

Thielman, H. P. Pathological functions. Proc. Iowa Acad. Sci. **59**, 338–343 (1952).

A function f defined on a neighborhood space X is said to be peculiar with respect to a point property P if there exists a partition of X into two subsets X_1 and X_2, each everywhere dense in X and such that the property P holds at every point of X_1 and fails to hold at every point of X_2. The author gives some examples of functions which are peculiar with respect to continuity and neighborliness (the last concept was introduced by W. W. Bledsoe [Proc. Amer. Math. Soc. **3**, 114–115 (1952); these Rev. **13**, 634]), and shows that there exist point properties with respect to which no function can be peculiar. One such property, named cliquishness, is defined as follows. Let the range of f be a metric space with metric ρ. f is said to be cliquish at a point c if given $\epsilon > 0$ and a neighborhood N of c there is a neighborhood $N_c \subset N$ (not necessarily containing c) such that $x \in N$, $x' \in N$ imply $\rho[f(x), f(x')] < \epsilon$. *M. Cotlar.*

Thielman, H. P. Types of functions. Amer. Math. Monthly **60**, 156–161 (1953).

This paper covers the same material as in the paper reviewed above. *M. Cotlar* (Buenos Aires).

Reifenberg, E. R. Parametric surfaces. III. The problem of Geöcze. Quart. J. Math., Oxford Ser. (2) **3**, 227–234 (1952).

Let S be any parametric surface (continuous image of a closed disc); let A be the class of sequences S_n of polyhedral surfaces inscribed in S with their boundaries inscribed in the boundary of S and tending to S in the sense of Fréchet. Let $L_1(S) = \text{Inf lim inf } a(S_n)$, where $a(S_n)$ denotes the elementary area of S_n, where lim inf is taken as $n \to \infty$, and Inf with respect to A. Then, if $L(S)$ denotes the Lebesgue area of S, we have $L(S) \leqq L_1(S)$. The aim of the present paper is to prove that $L(S) = L_1(S)$ for all surfaces.

The result improves the analogous one obtained by a different method by the reviewer, where the requirement that the boundaries of the surfaces S_n be inscribed in the boundary of S need be satisfied only if the boundary of S is rectifiable [Rivista Mat. Univ. Parma **1**, 207–227 (1950); these Rev. **12**, 86]. On the other hand the following remarks are necessary. (1) The proof draws heavily upon a series of previous papers which have aroused some criticism [Reifenberg, Proc. Cambridge Philos. Soc. **47**, 687–698 (1951); **48**, 46–69 (1952); these Rev. **14**, 363; Besicovitch, ibid. **44**, 313–334 (1948); **45**, 5–13, 14–23 (1949); J. London Math. Soc. **23**, 241–246 (1948); these Rev. **10**, 520, 521]. (2) No proof is given that the Lebesgue area L^* is an outer Carathéodory measure; yet this fact is true since the reviewer has proved it recently on the basis of the theory for Lebesgue area [Bull. Amer. Math. Soc. **57**, 465 (1951)]. (3) The proof —scattered in the papers quoted above—of the inequality involving length and Lebesgue area causes some doubts. An analogous inequality has been proved by the reviewer recently through a detailed analysis of the concepts involved [Bull. Amer. Math. Soc., **57**, 168 (1951)]. *L. Cesari.*

Reifenberg, E. R. Parametric surfaces. IV. The generalised Plateau problem. J. London Math. Soc. **27**, 448–456 (1952).

Let Λ^2 be the 2-dimensional Hausdorff measure over the Euclidean 3-space E_3; let $F(P)$ be a positive bounded lower semicontinuous function defined in E_3; let Γ be any continuous parametric contour. The object of this paper is to prove the theorem that the integral $I = \int_S F(P)d\Lambda^2$ attains its minimum in the class of all continuous parametric surfaces S subtending Γ. The author's procedure is based on a smoothing process which takes the place of a previous simpler one of A. S. Besicovitch concerning the analogous problem for the area ($F=1$), and on a lemma stating the lower semicontinuity of the integral $J = \int_S F(P)dL^*$, where L^* is the Lebesgue area considered as an outer Carathéodory measure. Such a lemma concerning J can be used in a proof for the existence of the minimum of I because of the known fact that for smooth surfaces $I = J$.

The reviewer has consistently used the standard property of lower semicontinuity of Lebesgue area in the calculus of variations (in particular, in surface area theory); nevertheless he could not follow the proof of the lemma, which draws heavily upon properties of the 2-dimensional Hausdorff measure and practically on the whole series of results of Reifenberg and Besicovitch [see list in the previous review]. The reviewer cannot conceal the impression that a clearer proof of the lemma could be given by using only Lebesgue area. With regard to the theorem it might be said that all proofs known to the reviewer for the Plateau and analogous problems suppose that Γ is a simple curve in E_3, while Reifenberg and Besicovitch appear to disregard this hypothesis.

L. Cesari (Lafayette, Ind.).

Albertoni, S., e Cugiani, M. Sul problema del cambiamento di variabili nella teoria delle distribuzioni. II. Nuovo Cimento (9) **10**, 157–173 (1953).

[For part I see Nuovo Cimento (9) **8**, 874–888 (1951); these Rev. **13**, 446.] This paper studies the effect of a change of variables on a (Schwartz) distribution on functions of several variables. The results are applied to prove some known formulas concerning Dirac's singular functions.

I. E. Segal (Chicago, Ill.).

Theory of Functions of Complex Variables

Simonart, Fernand. Sur les déplacements dans le plan complexe. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 885–891 (1952).

The author remarks that Carathéodory in his Funktionentheorie [Band I, Birkhäuser, Basel, 1950; these Rev. **12**, 248] bases the theory of displacements in the Gaussian plane on a study of the intersection of the Riemann sphere with planes passing through a fixed point. The author then shows how it is possible to describe these displacements in terms of the circular geometry of the Gaussian plane.

A. J. Lohwater (Ann Arbor, Mich.).

Valverde, Facundo. On the monogeneity of certain functions. Gaceta Mat. (1) **4**, 131–133 (1952). (Spanish)

Beckenbach, E. F. A property of mean values of an analytic function. Rend. Circ. Mat. Palermo (2) **1**, 157–163 (1952).

A theorem of Fejér and Riesz [Math. Z. **11**, 305–314 (1921)] says that if $f(z)$ is continuous in $|z| \leqq 1$ and analytic in $|z| < 1$, then $\pi^{-1}\int_{-1}^{1} |f(re^{i\theta})|\,dr \leqq (2\pi)^{-1}\int_0^{2\pi} |f(e^{i\theta})|\,d\theta$, equality being attained only if $f(z) = 0$. In the way of a converse, Nehari [C. R. Acad. Sci. Paris **208**, 1785–1787 (1939)] has shown that if $f(z)$ is analytic in $|z| < 1$, $f(0) = 0$, and if $\nu(\theta, f) = \int_0^1 |f(re^{i\theta})|\,dr \leqq 1$ for $0 \leqq \theta \leqq 2\pi$, then

$$\mu(r, f) = (2\pi)^{-1}\int_0^{2\pi} |f(re^{i\theta})|\,d\theta \leqq 1$$

for $r \leqq 1/2$, where $1/2$ is the largest value, as an example shows. Nehari has also shown that the hypothesis $f(0) = 0$ can be replaced by the condition that $[f(z)]^{1/2}$ be analytic in $|z| < 1$. The author is able to drop the condition that $[f(z)]^{1/2}$ be analytic in $|z| < 1$ and to describe the behavior of $\mu(r, f)$ for $1/2 < r < 1$. *A. J. Lohwater.*

Gahov, F. D. Riemann's boundary problem for a system of n pairs of functions. Uspehi Matem. Nauk (N.S.) **7**, no. 4(50), 3–54 (1952). (Russian)

In this comprehensive study the work on the Riemann boundary value problem for systems of analytic functions of a complex variable done by the author, by Mushelišvili, Vekua, Magnaradze and others [see, e.g., these Rev. **6**, 272; **10**, 439; **11**, 169] is summarized and extended in various directions. The basic problem is as follows: Let L be a system of disjoint closed smooth curves bounding a bounded region D^+ and the complementary region D^-. A "piecewise analytic" vector $\varphi(z)$ with components $\varphi_i^+(z)$ analytic in D^+ and $\varphi_i^-(z)$ analytic in D^- $(i = 1, 2, \cdots, n)$ is to be found such that $\varphi^+ = A\varphi^- + b$ on L. Here $\varphi^+(t)$, $\varphi^-(t)$ are the limits of $\varphi(z)$ as z approaches t on L from within D^+, D^- respectively, $A(t) = (a_{ij}(t))$ is a matrix and $b(t) = (b_i(t))$ a

vector of functions assigned on L, which satisfy a Hölder condition and are such that $\det A(t) \neq 0$ on L. Using the Cauchy integral, an equivalent system of Fredholm equations is obtained from which it can be deduced that the homogeneous boundary value problem ($b=0$) has n linearly independent solutions, which may have poles at ∞. From this is derived the fundamental result: The homogeneous problem has a "canonical system" of solutions $\varphi^1, \cdots, \varphi^n$ defined by the property that the matrix $(\varphi_i^j(z))$ is nowhere singular in the finite part of the plane and is of "normal form" at ∞. Between the solutions of a canonical system there can be no linear dependence with polynomial coefficients. From a canonical system the general solution of both the homogeneous and the nonhomogeneous problem is readily obtained. The results are generalized to the case where the $a_{ij}(t)$ become infinite of integral order and/or $\det A(t)$ vanishes at a finite number of points on L. Also the case is considered where the $a_{ij}(t)$ have jump discontinuities at a finite number of points on L. Other generalizations refer to a boundary L that consists of disjoint arcs, or that lies on an algebraic Riemann surface, or the problem may be generalized so that φ^+, φ^- refer to limits of $\varphi(z)$ at different points on L: $\varphi^+(\alpha(t)) = A(t)\varphi^-(t) + b(t)$, where $\alpha(t)$ transforms L biuniquely onto itself and has a nonvanishing derivative.

The vector $\omega(t) = \varphi^+(t) - \varphi^-(t)$ satisfies a system of singular integral equations of the form

$$C(t)\omega(t) + \frac{D(t)}{\pi i}\int_L \frac{\omega(\tau)}{\tau - t}d\tau + \int_L K(t, \tau)\omega(\tau)d\tau = g(t),$$

where the first integral is a Cauchy principal value and $K(t, \tau)$ is a regular kernel matrix (in the sense of Fredholm), and any such system leads to a Riemann boundary-value problem. More general integro-functional systems correspond to the generalized boundary value problems. If $\varphi(z)$ is a solution of the Riemann integro-differential boundary-value problem

$$\sum_{k=0}^{m}\left[a_k(t)\varphi^{+(k)}(t) + \int_L A_k(t, \tau)\varphi^{+(k)}(\tau)d\tau\right]$$

$$-\sum_{k=0}^{n}\left[b_k(t)\varphi^{-(k)}(t) + \int_L B_k(t, \tau)\varphi^{-(k)}(\tau)d\tau = f(t),\right.$$

then $\mu(t) = \varphi^{+(m)}(t) - t^n\varphi^{-(n)}(t)$ satisfies a singular integral system as above. A final generalization is obtained with the replacement of t in $\varphi_i^{(k)}(t)$ by $\alpha_{ik}(t)$, where $\alpha_{ik}(t)$ is a function like $\alpha(t)$ above. *M. Golomb* (Lafayette, Ind.).

Gahov, F. D. On inverse boundary problems. Doklady Akad. Nauk SSSR (N.S.) **86**, 649–652 (1952). (Russian)

Soient: L_w, la courbe fermée, simple, assez régulière du plan de la variable complexe $w = u + iv$, définie au moyen des équations paramétriques $u = u(s)$, $v = v(s)$, $0 \leqq s \leqq l$, où $u(s)$ et $v(s)$ admettent la période l; D_w^+, D_w^- les domaines intérieur et extérieur de w, limités par L_w; L_z, D_z^+, D_z^-, des éléments analogues du plan $z = x + iy$ (D_z^+ pouvant être multiplement connexe). L'A. se propose d'associer à L_w un contour L_z, de manière que $u(s) + iv(s)$ soit la valeur frontière de la fonction analytique, réalisant l'application conforme de D_z^+ (ou D_z^-) sur D_w^+ ou D_w^-. On constate que le problème intérieur a toujours une solution et une seule—moyennant, bien entendu, quelques hypothèses de régularité. Pour le problème intérieur, l'unicité n'a lieu que moyennant une condition supplémentaire dont la discussion reste à faire. *J. Kravtchenko* (Grenoble).

Iwata, Gïiti. Orthogonal functions in the complex domain. Progress Theoret. Physics **7**, 333–344 (1952).

The author seems mainly concerned with biorthogonal systems $f_r(z)$, $f_{r'}(z)$ regular respectively inside and on and outside and on a contour C, the inner product being $(f, g) = (1/2\pi i)\int_C f(z)g(z)dz$; formal matrix relations connect such a system with the system z^r, z^{-r-1}. In some cases there is a bilinear expansion of $(z'-z)^{-1}$. Examples are furnished by particular sets of eigenfunctions, associated with Bessel functions or functional equations such as $g(kw) = lg(w)$. A large number of examples, many associated with the classical polynomials, are then given independently of any eigenvalue problem. In the reviewer's opinion the topic is sufficiently interesting to merit re-writing from a rigorous point of view. *F. V. Atkinson* (Ibadan).

Walsh, J. L. Degree of approximation to functions on a Jordan curve. Trans. Amer. Math. Soc. **73**, 447–458 (1952).

Let $f(z)$, defined on an analytic Jordan curve C that contains the origin in its interior, be approximated on C by polynomials $p_n(z) = \sum_{k=-n}^{k=n} a_{nk}z^k$ $(n = 1, 2, \cdots)$. The main problem is the relation between the degrees of approximation of $p_n(z)$ to $f(z)$, and of $p_{jn}(z)$ to $f_j(z)$ $(j = 1, 2)$; here $p_{jn}(z)$ and $f_j(z)$ are the components of $p_n(z)$ and $f(z)$, respectively:

$$p_{1n} = \sum_{k=0}^{n} a_{nk}z^k, \quad p_{2n} = \sum_{k=-n}^{-1} a_{nk}z^k; \quad f_j(z) = (2\pi i)^{-1}\int_C (t-z)^{-1}f(t)dt,$$

with z inside C for $j=1$, outside for $j=2$. Under the conditions of the theorems, the $f_j(z)$ have boundary values on C. The author proves (Theorem 2) that, if on C

$$|f(z) - p_n(z)| \leqq A_3/n^{p+\alpha}$$

$$(0 < \alpha \leqq 1,\ p \text{ an integer} \geqq 0;\ n = 1, 2, \cdots)$$

then $|f_j(z) - p_{jn}(z)| \leqq A_3 \log n/n^{p+\alpha}$ for $j = 1, 2$; $n = 2, 3, \cdots$ on C, therefore also for every z not exterior $(j=1)$ or not interior $(j=2)$ to C. The constants A_i are independent of n. The hypothesis had been shown (Theorem 1) to be equivalent to the condition that $f^{(p)}(z)$ should belong to some Lipschitz or Zygmund class for $0 < \alpha < 1$ or $\alpha = 1$, respectively. Then approximation on C by analytic functions more general than the $p_n(z)$ is considered, e.g., in Theorem 4: Suppose that C be an analytic Jordan curve, the $F_n(z)$ rational functions of degree $n = 1, 2, \cdots$ whose poles have no limit point on C, and that $|f(z) - F_n(z)| \leqq A_3/n^{p+\alpha}$ on C. Then, in some annular region containing C, $|F_n(z)| \leqq A_7R^n$, where $R > 1$, and $|f_j(z) - F_{jn}(z)| \leqq A_9 \log n/n^{p+\alpha}$ on C, where $F_{jn}(z) = (2\pi i)^{-1}\int_C (t-z)^{-1}F_n(t)dt$, z interior $(j=1)$ or exterior $(j=2)$ to C. Finally the author deals with a multiply connected region. *H. Kober* (Birmingham).

Carleson, Lennart. On bounded analytic functions and closure problems. Ark. Mat. **2**, 283–291 (1952).

Let B be the space of bounded analytic functions with the uniform norm $\|f\| = \sup_{|z|<1} |f(z)|$; let C_μ be the space of functions f analytic in $|z| < 1$ and such that

$$\lim_{r \to 1} f(re^{i\theta})u(r) = 0,$$

uniformly in θ, with norm $\|f\|_\mu = \sup_{|z|<1} \mu(|z|)|f(z)|$. Here, μ may be any positive continuous function on $[0, 1]$ with $\mu(1) = 0$. As a Banach space, $C_\mu \supset B$. Theorem 2: Every functional L on C_μ, when restricted to B, has the form $L(f) = \int f(e^{i\theta})K(\theta)d\theta$ when $K \in L^1(0, 2\pi)$. Theorem 1: Every

functional L on C_a, when restricted to B, has the form $L(f)=\int f(e^{i\theta})\overline{g(e^{i\theta})}d\theta$ for some $g\in H^1$, if and only if $\limsup_{r\to 1}\mu(r)\log 1/(1-r)<\infty$. The first may be generalized to a result which characterizes weak closure in subspaces of $L^\infty[0, 2]$ by means of μ-closure in the corresponding space of harmonic functions obtained from L^∞ by the Poisson integral. Turning now to B as an algebra under pointwise multiplication, conditions are obtained under which the ideal generated by a set of functions f_α is all of B. Theorem 6: Let $E\subset B$ be such that for every $|a|<1$ there is an $f\in E$ with $f(a)\neq 0$, while for any $|a|=1$, there is an $f\in E$ with $\liminf_{z\to a}|f(z)|>0$. Then, the ideal generated by E is B itself. This has the following application for function theory. Theorem 7: Let $a_n=r_ne^{i\theta_n}$ with $r_n<1$ and $\sum(1-r_n)<\infty$. Let E be the closure of the set $\{\theta_1, \theta_2, \cdots\}$. Let $F(z)$ be analytic in $|z|<1$ with $\int\log^+|F(e^{i\theta})|d\theta$ bounded as $r\uparrow 1$, and suppose that $F(z)$ is bounded on the set $e^{i\theta}$ for $\theta\in E$. Then, there is a function f in B such that $f(a_n)=F(a_n)$. *R. C. Buck* (Madison, Wis.).

Combes, Jean. Sur les dérivées successives des fonctions analytiques. C. R. Acad. Sci. Paris **236**, 270–271 (1953).

The author announced results of the following nature. Let $f(z)$ be analytic in a region containing the origin, and let $\{n_k\}$ be a sequence of integers for which $\lim f^{(n_k)}(z)=F(z)$, uniformly in a neighborhood of the origin. Let $d_k=n_{k+1}-n_k$. In terms of n_k and d_k, the author discusses the nature of F and f. If $d_k\to\infty$, then F may be arbitrary; if this is not the case, then F obeys the differential equation $F^{(p)}=F$ for some p. If d_k is bounded, f is entire, of order ≤ 1. If d_k is unbounded, but $d_k/n_k\leq\alpha$, then f is entire of order $\leq 1+\alpha$. If $\lim f^{(n)}(z)=\infty$ on a neighborhood of the origin, and $f(z)=\sum a_nz^n/n!$, then $\lim|a_n|=\infty$; the converse holds if $|a_{n+1}/a_n|$ is bounded, but not if it is unbounded. The author also discusses the effect of hypotheses such as: $\lim f^{(2n)}(a)$ exists and $\lim f^{(2n+1)}(b)$ exists. *R. C. Buck.*

Combes, Jean. Sur les dérivées successives des fonctions analytiques. II. C. R. Acad. Sci. Paris **236**, 653–655 (1953).

Enlarging somewhat upon the paper reviewed above, the author observes that some results will carry over with little change to the study of $\lim C_nf^{(n)}(z)$ and $\lim_{n\to\infty}(1/\phi(n))\sum_0^n C_nf^{(n)}(z)$ where f is a fixed function, analytic at the origin. *R. C. Buck* (Madison, Wis.).

Sheffer, I. M. The derivatives of certain functions. J. Indian Math. Soc. (N.S.) **16**, 83–97 (1952).

L'auteur donne un théorème analogue à celui démontré par Ganapathy Iyer [même J. (N.S.) **8**, 94–108 (1944); ces Rev. **7**, 117] et généralisé par Boas et Chandrasekharan [Bull. Amer. Math. Soc. **54**, 523–526, 1191 (1948); ces Rev. **10**, 21, 287]. Soit $f(z)$ une fonction entière telle qu'en deux points z_1 et z_2, les suites $\{f^{(n)}(z_1)\}$, $\{f^{(n)}(z_2)\}$ n'aient qu'un nombre fini de points limites, tous à distance finie. Dans ces conditions on a $f(z)=E(z)+\phi(z)$, où $E(z)=\sum_{k=0}^{m-1}A_ke^{\omega^k z}$, avec $\omega=2\pi i/m$, où m est un entier convenablement choisi, et où $\phi(z)$ est une fonction entière telle que $\phi^{(n)}(z)\to 0$ uniformément sur tout ensemble borné. *S. Mandelbrojt.*

Buck, R. Creighton. On the distribution of the zeros of an entire function. J. Indian Math. Soc. (N.S.) **16** (1952), 147–149 (1953).

Es sei $n(r)$ die Anzahlfunktion einer ganzen Funktion vom Typus τ (>0) der Ordnung ρ (>0) und L und l der obere und untere Limes von $x^{-\rho}n(x)$. Die von R. P. Boas [Ann. of Math. (2) **47**, 21–32 (1946); diese Rev. **7**, 425] gegebenen Ungleichungen $l\leq\rho\tau$, $L\leq e\rho\tau$ werden vom Verf. durch die Ungleichung $l+eL\leq e^2\rho\tau$ ergänzt. *A. Pfluger.*

Boas, R. P., Jr. Inequalities between series and integrals involving entire functions. J. Indian Math. Soc. (N.S.) **16**, 127–135 (1952).

Es sei $f(z)$ eine ganze Funktion vom Typus c der Ordnung 1, $\phi(t)$ (≥ 0) eine nicht abnehmende konvexe Funktion von $\log t$ und $\{\lambda_n\}$ eine reelle Zahlenfolge mit $\lambda_{n+1}-\lambda_n\geq 2\delta>0$. Verf. beweist die Ungleichung

$$\frac{\pi\delta}{2}\sum_{-\infty}^{+\infty}\phi(e^{-c\delta}|f(\lambda_n)|)\leq\int_{-\infty}^{\infty}\phi(|f(x)|)dx.$$

Diese verallgemeinert und verschärft Resultate von Plancherel und Pólya [Comment. Math. Helv. **9**, 224–248 (1937); **10**, 110–163 (1938)] mit $\phi=t^p$, $p>0$; wobei in diesem Falle der Faktor $e^{-c\delta}$ weggelassen werden kann. Auch für Funktionen, die in einer Halbebene analytisch und vom Normaltypus der Ordnung 1 sind, erhält Verf. gewisse Resultate. *A. Pfluger* (Zürich).

Schaeffer, A. C. Entire functions and trigonometric polynomials. Duke Math. J. **20**, 77–88 (1953).

In this review $f(z)$ stands for an entire function of exponential type: $f(z)=O(e^{\gamma|z|})$ for some $\gamma\geq 0$. S. Bernstein has shown that the hypothesis $|f(x)|\leq 1$ for $-\infty<x<\infty$ implies that $|f'(x)|\leq\gamma$ for $-\infty<x<\infty$ [C. R. Acad. Sci. Paris **176**, 1603–1605 (1923)]. The present author assumes only that $|f(x)|\leq 1$ holds on a closed subset E of $-\infty<x<\infty$ with the following property. There are positive numbers A and α such that every interval of length A contains a subset of E of measure greater than α. The author proves that for almost all x_0 of E there exists a number c depending on x_0 and E, but not on $f(z)$ or γ, such that $|f'(x_0)|\leq\gamma c$. The proof is as follows. The author constructs a single-valued function $u(x, y)$, harmonic outside E, having the limit 0 at almost all points of E, and such that $u(x, y)-|y|$ is bounded and nonnegative, while $u_y(x, y)>0$ for $y>0$, $u_y(x, y)<0$ for $y<0$. Next a result of Carathéodory-Nevanlinna on functions with positive real part is brought in to show that for almost all x_0 of E there is a number $p=p(x_0)$ such that $u(x, y)\leq p|y|$ for $|x-x_0|\leq|y|$. It follows that the function $F(z)=e^{u+iv}$, where v is the harmonic conjugate of u, satisfies the inequality

$$|F(z)|\leq\exp(p|z-x_0|).$$

It is then shown that the function $f(z)$ satisfies the estimate $|f(z)|\leq|F(z)|^\gamma$, and the desired result follows.

The author's result furnishes a proof for the following conjecture by Privaloff [ibid. **162**, 123–126 (1916)]. Let H be a closed subset of $-\pi\leq x\leq\pi$ of positive measure $|H|$, and let the trigonometric polynomial $T_n(x)$ of order n satisfy $|T_n(x)|\leq 1$ for x in H. Then for every $\epsilon>0$ there are a subset H^Δ of H of measure greater than $|H|-\epsilon$ and a number $B(\epsilon)$ such that $|T_n'(x)|\leq nB(\epsilon)$ for x in H^Δ. This result may be used in the study of conjugate trigonometric series. *J. Korevaar* (Madison, Wis.).

Shah, S. M., and Ishaq, M. On the maximum modulus and the coefficients of an entire series. J. Indian Math. Soc. (N.S.) **16** (1952), 177–182 (1953).

Let $f(z)=\sum a_nz^n$ be an entire function, $M(r)$ its maximum modulus, $\mu(r)$ the maximum term in the series and $\nu(r)$ the rank of this term. Let l_kr denote the kth iterate of $\log r$. The authors establish a large number of equalities and in-

equalities among the lim sup and lim inf of $l_k M(r)$, $l_k\mu(r)$, or $l_{k-1}\nu(r)$, divided by $l_k r$ or $l_{k-1}r$; and of $l_k n$ or $l_{k+1}n$ divided by $l_k(n^{-1}\log|a_n|^{-1})$. Some typical results are as follows: $t_k=\liminf\{l_kM(r)\}/l_kr=\liminf\{l_k\mu(r)\}/l_kr$, $k\geqq 2$; the same with lim sup; if $\delta_k=\liminf\{l_{k-1}\nu(r)\}/l_kr$, then $t_2=1+\delta_2$, while $t_k=\max(1,\delta_k)$, $k\geqq 3$; correspondingly with lim sup.

R. P. Boas, Jr. (Evanston, Ill.).

Tricomi, Francesco G. Una nuova funzione introdotta dalla batteriologia. Univ. e Politecnico Torino. Rend. Sem. Mat. 11, 35–46 (1952).

In this lecture the author discusses the entire function

$$G(z)=\sum_{n=1}^{\infty}\frac{1}{a^n-1}\frac{z^n}{n!},\quad a>1,$$

giving more details than in an earlier paper [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 11, 141–144 (1951); these Rev. 13, 637]. In particular, he now discusses the behavior of $G(z)$ as $z\to\infty$ in the left half-plane, and describes the periodic properties of $K(u)+\Phi(u)$ [the notations being those explained in the cited review].

A. Erdélyi (Pasadena, Calif.).

Bazilevič, I. E., and Korickiĭ, G. V. On some properties of univalent conformal mappings. Mat. Sbornik N.S. 32(74), 209–218 (1953). (Russian)

Let $f(z)$ be univalent in $|z|<1$, and $f(0)=0$. It is known that if $f(z)$ maps $|z|\leqq r$ onto a convex region; the same is true for $|z|\leqq\rho$ for each $\rho<r$. A similar assertion holds for starlike domains. The authors show that a "natural" extension of these two results is not possible. They prove: (I) For each integer $N\geqq 1$, there is a function $f(z)$ univalent in $|z|<1$, and two numbers $\rho_1<\rho_2<1$ such that the image curve under $f(z)$ of $|z|=\rho_1$ has at least $4N$ inflection points, and for each ρ, $\rho_2\leqq\rho<1$, the image curve of $|z|=\rho$ has only two inflection points. (II) There is a function $F(z)$, univalent in $|z|<1$, and two numbers $\rho_1<\rho_2<1$, such that $\partial\arg F(z)/\partial\varphi$ changes sign $4N$ times on the circle $|z|=\rho_1$ and on each circle $|z|=\rho$, $\rho_2\leqq\rho<1$, it changes sign only 4 times.

A. W. Goodman (Lexington, Ky.).

***Beckenbach, E. F., and Graham, E. W. On subordination in complex variable theory.** Construction and applications of conformal maps. Proceedings of a symposium, pp. 247–254. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

The authors define a system of functions $f_1(z)$, $f_2(z)$, $\cdots$, $f_n(z)$ to be joint subordinate to $f(z)$ under T in $|z|<r$, where $T(w_1,w_2,\cdots,w_n)$ is a regular analytic function of $w_1,\cdots,w_n$ in the range of values given by $w_j=f_j(z)$, $|z|<r$, if for each set of complex constants α_j, $j=1,\cdots,n$, with $|\alpha_j|\leqq 1$, $T[f_1(\alpha_1 z),\cdots,f_n(\alpha_n z)]$ is subordinate to $f(z)$. They prove that if $f_1(z)$, $f_2(z)$, $\cdots$, $f_n(z)$ are joint subordinate to $f(z)$ under T in $|z|<1$ then also in $|z|<r$ for $0<r<1$. A consequence is the following main theorem. Suppose $w=f(z)$ maps $|z|<1$ one-to-one conformally onto D with $f(0)=0$. Let D_r be the image of $|z|<r$ under the map. Then if for w_j in D_{r_j}, $0\leqq r_j\leqq 1$, $T(w_1,w_2,\cdots,w_n)$ always lies in D_{r_0}, then for w_j in $D_{r_j r}$, $T(w_1,\cdots,w_n)$ lies in $D_{r_0 r}$. The case $r_1=r_2=\cdots=r_n=1$ is due to Ford [Duke Math. J. 1, 103–104 (1935)]. A variety of applications are obtained very simply of which the following is an example. If D_1 contains the convex hull of D_{r_1}, then D_r contains the convex hull of $D_{r_1 r}$ for $0\leqq r\leqq 1$. An application to cartography is also given.

W. K. Hayman (Exeter).

***Shiffman, Max. On the effective determination of conformal mapping.** Construction and applications of conformal maps. Proceedings of a symposium, pp. 227–229. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

It is pointed out that since, from a practical viewpoint, the principal use of conformal mapping consists in reducing a boundary-value problem of potential theory for a given domain to a corresponding problem for a simpler domain like a circle or a half-plane, we do not need so much to study the details of the conformal mapping everywhere but rather might restrict our attention to the correspondence established between the boundaries. Thus it would seem desirable to have a proof of the Riemann mapping theorem which emphasizes the boundary of the region rather than the region itself.

The problem of mapping the interior of the unit circle conformally on the interior of a plane simply connected domain is the problem of determining functions $x(u,v)$, $y(u,v)$ which assume the desired boundary values and minimize the Dirichlet functional

$$D=\frac{1}{2}\int\int[(x_u^2+x_v^2)+(y_u^2+y_v^2)]\,du\,dv.$$

An expression for this functional of harmonic functions in terms of given boundary values has been found by Douglas [Trans. Amer. Math. Soc. 33, 263–321 (1931)]; it is

$$D=\frac{1}{4\pi}\int\int\frac{[x(\theta)-x(\varphi)]^2+[y(\theta)-y(\varphi)]^2}{4\sin^2(\theta-\varphi)}\,d\theta\,d\varphi.$$

An alternative expression found by the author [Ann. of Math. (2) 40, 834–854 (1939); these Rev. 1, 79] is:

$$D=\frac{1}{4\pi}\int\int\log\frac{1}{4\sin^2\frac{1}{2}(\theta-\varphi)}[dx(\theta)dx(\varphi)+dy(\theta)dy(\varphi)];$$

this has the advantage that if θ is replaced by

$$(1-t)\lambda_0(\theta)+t\lambda_1(\theta),\quad 0\leqq t\leqq 1,$$

where $\lambda_0(\theta)$ and $\lambda_1(\theta)$ are monotonic functions of θ, then the integrand becomes a convex function of t.

E. F. Beckenbach (Los Angeles, Calif.).

***Courant, R. Flow patterns and conformal mapping of domains of higher topological structure.** Construction and applications of conformal maps. Proceedings of a symposium, pp. 7–14. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

The author reviews the relationship between certain flow patterns and the canonical conformal mapping of Riemann surfaces onto parallel-slit domains. The discussion is carried through for the case of surfaces of higher genus, whether orientable or not, by the introduction of slit domains of several sheets.

P. R. Garabedian (Stanford, Calif.).

Bergman, Stefan. A theorem of Green's type for functions of two complex variables. Proc. Amer. Math. Soc. 4, 102–109 (1953).

For functions of the extended class, the author develops an integration formula relating domain integrals with integrals over the distinguished boundary surface. The results are applied to obtain estimates of integrals of analytic functions of two complex variables.

P. R. Garabedian.

Hervé, Michel. Sur les fonctions fuchsiennes de deux variables complexes. Ann. Sci. Ecole Norm. Sup. (3) 69, 277–302 (1952).

The theory of Fuchsian functions of two complex variables has been studied by the author and the principal results were published without proofs in three notes [C. R. Acad. Sci. Paris **226**, 462–464 (1948); **232**, 673–675 (1951); **234**, 41–43 (1952); these Rev. **9**, 343; **12**, 693; **13**, 645]. The finer results were stated only for Fuchsian functions in the bicylinder $|x|<1$, $|y|<1$. In the present paper the author gives a detailed account of the theory in a general domain and the statements from the three notes are proved. The Fuchsian functions in a domain D are attached to a discontinuous group of analytic automorphisms of D, and it is an open question whether such groups exist in any domain that cannot be mapped analytically on a bicylinder or a sphere. *H. Tornehave* (Lyngby).

Hua, Loo-Keng. Theory of automorphic functions of several complex variables. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **15**, 243–273 (1947). (English. Georgian summary)

This paper is virtually coextensive and synchronous with an earlier paper of the author [Ann. of Math. (2) **47**, 167–191 (1946); these Rev. **7**, 429], except that the present paper is somewhat more detailed in illustrating the general statements by the more particular cases of matrix spaces. [In the meantime statements of these papers have been considerably generalized to manifolds instead of ordinary domains, some by this reviewer.]

S. Bochner (Princeton, N. J.).

Sampson, J. H. A note on automorphic varieties. Proc. Nat. Acad. Sci. U. S. A. **38**, 895–898 (1952).

Soit D un domaine borné de l'espace numérique complexe et Γ un groupe discret d'homéomorphismes analytiques de D. Toute fonction holomorphe $\Theta(Z)$ satisfaisant, pour $Z \in D$ et $\gamma_i \in \Gamma$ à $\Theta(\gamma_i Z)=\Theta(Z)\cdot[J_i(Z)]^{-k}$ où J_i désigne le jacobien de γ_i, est dite forme automorphe de poids k [C. L. Siegel, Analytic functions of several complex variables, Inst. Advanced Study, Princeton, 1950; ces Rev. **11**, 651]. Théorème 1: La dimension de l'espace vectoriel des formes automorphes de poids k croît indéfiniment avec k. La démonstration repose sur la propriété suivante: soit A un point donné de D non fixe pour les éléments de Γ différents de l'élément neutre; il existe une série de Poincaré de poids k suffisamment grand, dont les valeurs en A des dérivées partielles d'ordre inférieur à un nombre donné arbitraire différent de moins de ϵ de nombres complexes arbitrairement donnés. Soit M la variété analytique complexe $M=D \pmod{\Gamma}$ supposée compacte; M est appelée variété automorphe. Le groupe G des homéomorphismes analytiques de M est un groupe de Lie complexe, compact [Bochner and Montgomery, Ann. of Math. (2) **48**, 659–669 (1947); ces Rev. **9**, 174]. Les densités holomorphes de poids k sur M coïncident avec les projections des formes automorphes de poids k sur D. La transformée d'une densité sur M par un élément de G est une forme linéaire des densités sur M dont les coefficients sont des fonctions holomorphes sur G, i.e., des constantes. Alors, si G est de dimension strictement positive, chaque densité définit une relation linéaire et homogène entre les générateurs de G à l'origine, d'où une contradiction à cause du théorème 1. Il en résulte Théorème 2: G est un groupe fini. *P. Dolbeault* (Paris).

Kryloff, N. M. Sur les transformations géométriques et les généralisations des fonctions d'une variable complexe. Soobščeniya Akad. Nauk Gruzin. SSR. **8**, 3–5 (1947). (Russian. French summary)

The author observes that a function $u(x, y)+\epsilon v(x, y)$ where x, y are real and $\epsilon^2=y\epsilon+\mu$ (y, μ real) will be a monogenic function of $x+\epsilon y$ if $u_y=\mu v_x$, $u_x=v_y-yv_x$ and comments on these generalized Cauchy-Riemann equations. *L. Bers.*

Nisigaki, Hisami. A theory of quaternion-functions, by classification. I. Mem. Fac. Sci. Eng. Waseda Univ. no. **15**, 6–14 (1951).

Nisigaki, Hisami. A theory of quaternion-functions, by classifications. II. Mem. Fac. Sci. Eng. Waseda Univ. no. **16**, 122–126 (1952).

Let $e_0=1$, e_2, e_3, e_4 be the unit quaternions and set $dX=\sum dx_i e_i$, $Y=\sum y_i(x_1, x_2, x_3, x_4)e_i$ where the x_i, y_i are real. The author calls the function $Y=Y(X)$ of class C_r if $dX=\sum_{\iota=1}^{r} A^{(\iota)}dXB^{(\iota)}$. He shows that every (smooth) function is of class C_4, that the functions of class C_3 are of the form $Y=UV^{-1}$ where U and V are right-regular functions of X in the sense of Fueter [R. Fueter, Comment. Math. Helv. **7**, 307–330 (1935); **8**, 371–378 (1936)], and that functions of class C_2 can be expressed in terms of analytic functions of two complex variables. Functions of class C_1 are linear fractional transformations [cf., for instance, the author's papers in Tôhoku Math. J. **45**, 73–102 (1939); **47**, 217–236 (1940); these Rev. **2**, 277].

L. Bers (New York, N. Y.).

*Isaacs, Rufus. **Monodiffric functions. Construction and applications of conformal maps.** Proceedings of a symposium, pp. 257–266. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

A study of the properties of functions, in general polygenic, which satisfy a difference equation relation analogous to the Cauchy-Riemann differential equations.

Philip Franklin (Cambridge, Mass.).

Theory of Series

Fain, Bill W. Evaluation of certain classes of infinite numerical series in closed form. Math. Mag. **26**, 121–126 (1953).

Some series of the form $\sum a_n b_n$ are evaluated by equating the coefficients in the Fourier product of the Fourier series of $f(x)$ and $g(x)$ to the coefficients in the Fourier series of $f(x)g(x)$. For example, taking $f(x)=g(x)=x^2$ gives, when k is a positive integer,

$$\sum_{n=1}^{\infty}\frac{1}{n^2(k+n)^2}=\frac{k^2\pi^2-9}{3k^4}-\frac{1}{2}\sum_{n=1}^{k-1}\frac{1}{n^2(k-n)^2}$$

R. P. Agnew (Ithaca, N. Y.).

Rajagopal, C. T. Sui criteri del rapporto per la convergenza delle serie a termini positivi. Boll. Un. Mat. Ital. (3) **7**, 382–387 (1952).

By use of the integral test for convergence of series the author obtains several familiar criteria, and modifications of these criteria, for convergence of series $\sum u_n$ of positive terms. The criteria involve $R_n=u_{n+1}/u_n$, $\log R_n$, and R_{n+1}/R_n. *R. P. Agnew* (Ithaca, N. Y.).

Basu, S. K. A note on the oscillation of the Cesàro and Hölder means of a sequence and a function. Bull. Calcutta Math. Soc. **44**, 45–50 (1952).

Let $H_n^{(\alpha)}$ and $C_n^{(\alpha)}$ denote the Hölder and Cesàro transforms of order α of a real sequence s_n. If $0<\alpha<1$, then the inequalities

$$(1)\quad \liminf_{n\to\infty} U_n \leqq \liminf_{n\to\infty} V_n \leqq \limsup_{n\to\infty} V_n \leqq \limsup_{n\to\infty} U_n$$

hold when $U_n=H_n^{(\alpha)}$ and $V_n=H_n^{(\alpha)}$. If $-1<\alpha<0$ or $\alpha>1$, then (1) holds when $U_n=C_n^{(\alpha)}$ and $V_n=H_n^{(\alpha)}$. If $1<\alpha<\beta$, then (1) fails when $U_n=H_n^{(\alpha)}$ and $V_n=C_n^{(\beta)}$. Similar theorems are proved for Hölder and Cesàro kernel transforms of functions $s(x)$. The results, which are well known when α is an integer, are proved very simply without this restriction by use of the Hurwitz-Silverman-Hausdorff theory of transformations that commute with $C^{(1)}$.

R. P. Agnew (Ithaca, N. Y.).

Chadaia, T. The mutual relation between Cesàro and Nörlund methods of summation of double series. Akad. Nauk Gruzin. SSR. Trudy Mat. Inst. Razmadze **18**, 237–244 (1951). (Georgian. Russian summary)

The Nörlund double sequence transformation considered is

$$W_{mn}=P_{mn}^{-1}\sum_{j=0}^{m}\sum_{k=0}^{n}p_{m-j,n-k}S_{j,k}$$

where $p_{j,k}>0$ and $P_{mn}=\sum_{j=0}^{m}\sum_{k=0}^{n}p_{jk}$. In case $p_{jk}=1$ when $j,\ k\geqq 0$, this becomes the Cesàro transformation and $W_{m,n}$ is denoted by $\sigma_{m,n}$. It is assumed that the sequence $p_{j,k}$ satisfies several conditions, one of them requiring that

$$p_{j,k}-p_{j+1,k}-p_{j,k+1}+p_{j+1,k+1}\geqq 0.$$

It is assumed that $S_{j,k}$ is a sequence for which the Cesàro transform $\sigma_{m,n}$ converges restrictedly to S in the sense of C. N. Moore, and that $\lim_{n\to\infty}\sigma_{m,n}/P_{m,n}=0$ and $\lim_{m\to\infty}\sigma_{m,n}/P_{m,n}=0$ for each n and m respectively. It is then shown that the Nörlund transform $W_{m,n}$ converges restrictedly to S.

R. P. Agnew (Ithaca, N. Y.).

Knopp, Konrad. Einige Bemerkungen zur A-, E_k- und B_k-Summierung. Rend. Circ. Mat. Palermo (2) **1**, 129–138 (1952).

A series $\sum a_n$ is evaluable to s by the Abel method A if the series $\sum a_n x^n$ converges over $0<x<1$ to a function $f(x)$ such that $f(x)\to s$ as $x\to 1$, and is absolutely evaluable A, or evaluable $|A|$ to s if in addition $f(s)$ has bounded variation over $0<x<1$. Similar definitions apply to the Euler-Knopp methods E_k and Borel-Sannia methods B_k of real nonnegative orders k. The methods B_0 and B_1 are the Borel exponential method and the Borel integral method respectively. It is shown that while a series $\sum u_n$ may be evaluable $|E_k|$ or $|B_k|$ without being evaluable $|A|$, this cannot happen if the series $\sum a_n x^n$ converges when $0<x<1$. This supplements the classic fact that, so far as application to series in the working field of the Abel method is concerned, the Euler-Knopp and Borel-Sannia methods are weaker than the Abel method. There is a brief historical account.

R. P. Agnew (Ithaca, N. Y.).

Vermes, P. Note on γ-matrices efficient at an isolated point. Proc. Edinburgh Math. Soc. (2) **10**, 11–12 (1953).

The result proved in this paper for a binomial series at a point z, $|z|>1$, holds for any series with unbounded partial sums [Darevsky, Izvestiya Akad. Nauk SSSR, Ser. Mat. **10**, 97–104 (1946); these Rev. **7**, 517]. *G. G. Lorentz.*

Lorentz, G. G., and Macphail, M. S. Unbounded operators and a theorem of A. Robinson. Trans. Roy. Soc. Canada. Sect. III. (3) **46**, 33–37 (1952).

The theorem of Robinson [Proc. London Math. Soc. (2) **52**, 132–160 (1950); these Rev. **12**, 253] is that which extends the Toeplitz criteria for regularity from transformations of sequences of complex numbers to transformations of sequences of elements of a general Banach space B. The theorem of Robinson is not a consequence of the familiar Hahn-Banach-Steinhaus theory of sequences of linear bounded operators. The authors give a theorem on linear, but possibly unbounded, operators T_m and deduce Robinson's theorem from it. Let $L_1, L_2, L_3, \cdots$ be closed linear subspaces of B. For each $m=1, 2, \cdots$, let T_m be a linear operator mapping B onto itself and bounded over L_m. For each x in B, let $T_1(x), T_2(x), \cdots$ be a bounded sequence in B. Then there is an integer N such that each T_m is bounded over the intersection $L_1L_2\cdots L_N$. Variants of this theorem and of Robinson's theorem are given.

R. P. Angew (Ithaca, N. Y.).

Macphail, M. S. The extended Euler-Knopp transformation. Trans. Roy. Soc. Canada. Sect. III. (3) **46**, 39–43 (1952).

Each suitable function $g(t)=\sum_{n=0}^{\infty}c_n t^n$ defines a method G for evaluation of series $\sum a_n$ as follows. Determine constants $b_0, b_1, \cdots$ such that $\sum b_n t^n=\sum a\{g(t)\}^k$ is a formal identity in t; if $\sum b_n$ converges to s, then $\sum a_n$ is evaluable G to s. The author supplements known facts about methods of the form G belonging to rather general classes. Questions considered are evaluability of power series, consistency, and evaluability of Cauchy product series. *R. P. Agnew.*

Delange, Hubert. Encore une nouvelle démonstration du théorème taubérien de Littlewood. Bull. Sci. Math. (2) **76**, 179–189 (1952).

Let $\sum a_n e^{-\lambda_n s}$ converge when $\mathrm{Re}(s)>0$ to a function $f(s)$. The author gives new proofs of theorems saying that if $f(s)\to S$ as $s\to 0$ over an appropriate set, and if $\sum a_n$ satisfies the Tauberian condition $|a_n|<M(\lambda_n-\lambda_{n-1})/\lambda_n$ or a more general Tauberian condition of Schmidt type, then $\sum a_n$ converges to S. The proof depends upon a Tauberian theorem for Laplace integrals. Proof of the latter theorem depends upon a theorem of Montel on the behavior of functions analytic in a sector of the complex plane.

R. P. Agnew (Ithaca, N. Y.).

Bohr, Harald. On the summability function and the order function of Dirichlet series. Danske Vid. Selsk. Mat.-Fys. Medd. **27**, no. 4, 39 pp. (1952).

Soit $f(s)=\sum a_n n^{-s}$ une série de Dirichlet ordinaire qui admet une abscisse de convergence finie ($-\infty<\lambda_0<\infty$). A tout $0\leqq r<\infty$ correspond un λ_r qui est l'abscisse de sommabilité d'ordre r. On sait que la fonction $\sigma=\lambda_r$ est non croissante, continue et convexe, avec une pente numérique $\leqq 1$. Posons $\Omega=\lim_{r\to\infty}\lambda_r$ ($\Omega\geqq-\infty$). Pour $\Omega<\sigma<\infty$, définissons la fonction $r=\psi(\sigma)$ comme étant la borne inférieure des valeurs $\rho\geqq 0$ pour lesquelles on a $\lambda_\rho\leqq\sigma$; $\psi(\sigma)$ est la fonction de sommabilité de la série de Dirichlet. $\mu(\sigma)$ étant la fonction d'ordre de Lindelöf, on sait que (1) $\psi(\sigma)\leqq u(\sigma)\leqq\psi(\sigma)+1$. L'auteur se pose la question suivante: deux fonctions $\psi(\sigma)$ et $\mu(\sigma)$ possédant les propriétés mentionnées plus haut ($\mu(\sigma)$ est continue et convexe, d'après Lindelöf) étant données, existe-t-il toujours une série de Dirichlet ordinaire pour laquelle $\psi(\sigma)$ et $\mu(\sigma)$ sont respectivement la fonction de sommabilité et la fonction d'ordre? La

réponse est affirmative si l'on ajoute que $\mu(\sigma)$ est de pente numérique (dans l'intervalle où elle décroit strictement) $\geqq 1$. Voici l'énoncé du théorème: Soit $\psi(\sigma)$ une fonction continue convexe définie pour $\sigma > \Omega (\geqq -\infty)$, avec $\psi(\sigma)=0$ pour $\sigma > \omega_\psi$, ω_ψ fini, $\omega_\psi \geqq \Omega$, et supposons que (si $\omega_\psi > \Omega$): $\psi'(\omega_\psi - 0) \leqq -1$. Supposons que $\mu(\sigma)$ est une fonction continue, convexe pour $\sigma > \omega$, avec $\mu(\sigma)=0$ pour ω_μ, ou ω_μ est fini, $\omega_\mu \geqq \Omega$, et supposons que (si $\omega_\mu > \omega$): $\mu'(\omega_\mu - 0) \leqq -1$. Supposons, enfin, que la relation (1) a lieu pour $\sigma > \Omega$. Il existe alors une série de Dirchlet ordinaire pour laquelle $\psi(\sigma)$ et $\mu(\sigma)$ sont respectivement la fonction de sommabilité et la fonction d'ordre. On ne sait pas si une fonction d'ordre doit nécessairement satisfaire à la condition (2) $\mu'(\omega_\mu - 0) \leqq -1$; appliquée à la fonction $\zeta(s)(1-2^{1-s})$, cette inégalité équivaut à l'hypothèse de Lindelöf, c'est-à-dire, $\mu(\frac{1}{2})=0$. *S. Mandelbrojt* (Paris).

Frank, Evelyn. On the properties of certain continued fractions. Proc. Amer. Math. Soc. **3**, 921–937 (1952).

The linear fractional transformation

$$s(t)=[a-zt]/[1-\bar{a}zt],$$

where z is a complex parameter and a a complex number with modulus less than unity, was used by I. Schur as the basis for his investigations of functions regular and with moduli less than 1 in the unit circle. The reviewer [Analytic theory of continued fractions, Van Nostrand, New York, 1948, Chap. XV; these Rev. **10**, 32] investigated the continued fraction arising from the "continued fraction-like algorithm" of Schur. The author investigates continued fraction expansions for power series, obtained by means of the same algorithm, but without requiring $|a|<1$ and introducing a new parameter at any step where the expansion process leads to $|a|=1$. A device for obtaining the expansions, based on an idea of the reviewer [loc. cit., pp. 196–200], is obtained. Conditions for convergence are given which are, in the main, applications of well-known convergence theorems. A theorem formally analogous to one of the reviewer [cf. loc. cit., pp. 288–292], but differing essentially with respect to applicability of the convergence conditions, is given. A bound for the value of the continued fraction is obtained in a particular case [cf. Wall, loc. cit., p. 42]. *H. S. Wall* (Austin, Tex.).

Fourier Series and Generalizations, Integral Transforms

Steinberg, R., and Redheffer, R. M. Simultaneous trigonometric approximation. J. Math. Physics **31**, 260–266 (1953).

The problem considered arose in electrical network theory. Let $F(z)$ be an arbitrary function defined on the unit circle. The problem is the r.m.s. approximation of F on the arc $(-\theta, \theta)$ of the unit circle by a polynomial $A_n(z)$ having real coefficients. The minimization of the error leads to formulas involving the positive definite matrix P with matrix elements $\sin(j-k)\theta/(j-k)$ for $j, k=0, 1, \cdots, n$. If $\theta<\pi$, the error vanishes as the degree n approaches infinity. The problem is also studied under the additional constraint that the sum of the squares of the coefficients of A_n shall not exceed a certain constant M. This constraint is essential for the network interpretation. It is shown that the diagonalization of P leads to a practicable method of determining the minimizing coefficients. *R. J. Duffin.*

Lorch, Lee. Asymptotic expressions for some integrals which include certain Lebesgue and Fejér constants. Duke Math. J. **20**, 89–103 (1953).

If $g(t)$ is integrable in the interval $(0, \pi)$ and is periodic with period π, and if the integral $\int_0^1 (g(t)/t)dt$ exists in some sense, then, for $0<b<\pi$,

$$(*)\quad \frac{1}{b}\int_0^b \frac{g((2x+1)t)}{\sin t}dt = \frac{m}{b}\log x + \alpha + o\left(\frac{1}{x}\right) \quad (x\to\infty),$$

where $m=\pi^{-1}\int_0^\pi g(t)dt$ and

$$\alpha = \frac{2m}{b}\log 2 + \frac{m}{b}\log\tan\frac{b}{2} + \frac{1}{b}\int_0^1 \frac{g(t)}{t}dt - \frac{1}{b}\int_1^\infty (m-g(t))\frac{dt}{t}.$$

In particular, taking $b=\pi/2$, $g(t)=|\sin t|$, x an integer n, (*) becomes the formula for the Lebesgue constant due to the author [Bull. Calcutta Math. Soc. **37**, 5–8 (1945); these Rev. **7**, 59]; and taking $g(t)=\sin^2 t$, (*) becomes the formula for the Fejér-Lebesgue constant due to Gronwall [Math. Ann. **72**, 244–261 (1912)]. The author gives further asymptotic formulas of similar integrals which give, as a special case, the formulas for the Borel-Lebesgue constant and the Euler-Lebesgue constant due to Hardy and Littlewood [ibid. **97**, 159–209 (1926)], and the author [Duke Math. J. **11**, 459–467 (1944); **19**, 45–50 (1952); these Rev. **6**, 48, **13**; 645]. *S. Izumi* (Tokyo).

Hsiang, Fu Cheng. On the integro-jump of a function and its Fourier coefficients. Bull. Calcutta Math. Soc. **44**, 55–58 (1952).

Let $\bar{s}_n(x)$ represent the nth Fejér mean of the series conjugate to the Fourier series of the Lebesgue integrable function $f(t)$. Let $\psi(x,t)=f(x+t)-f(x-t)-D(x)$. If $D(x)$ is such that the integral $\int_0^t \psi(x,u)u^{-1-\eta}du$ exists for some $\eta>0$, or alternatively if $\int_0^t\int_0^u \psi(x,u)dudt=o(t/\log(1/t))$ as $t\to 0+$, then

$$(*)\qquad \lim_{n\to\infty} \bar{s}_{2n}(x)-\bar{s}_n(x)=\pi^{-1}\log 2D(x).$$

Other conditions for the existence of the limit (*) were given by Szász [Duke Math. J. **4**, 401–407 (1938)]. *P. Civin.*

Džvaršeišvili, A. G. On a criterion of convergence of a Fourier series. Soobščeniya Akad. Nauk Gruzin. SSR **11**, 403–407 (1950). (Russian)

Suppose $f(x)$ is an integrable function over the interval $(-\pi, \pi)$ vanishing on a closed subset E. It is shown that the Fourier series of $f(x)$ converges to zero at every point of density of the set E provided

$$(*)\qquad \sum_{k=1}^{\infty}\omega(f,\delta_k)<+\infty$$

where the δ_k are the intervals contiguous to E in $(-\pi, \pi)$ and $\omega(f,\delta)$ is the oscillation of $f(x)$ over the interval δ. This result is used to prove that if $f(x)$ is bounded on E, satisfies (*), and has finite derivates at the points of density of E, and if $g(x)$ is integrable on $(-\pi, \pi)$ and is bounded on the complement of E, then the Fourier series of the functions $g(x)f(x)$ and $g(x)f(x_0)$ are equiconvergent at every point x_0 of density of E. *G. Klein* (South Hadley, Mass.).

Džvaršeišvili, A. G. On the representation by singular integrals of functions integrable in the sense of Denjoy-Perron. Soobščeniya Akad. Nauk Gruzin. SSR **11**, 473–478 (1950). (Russian)

A function $\phi_n(t,x)$ $(a\leqq x\leqq b, a<t<b, n=1, 2, \cdots)$ is said to be a kernel if $\lim_{n\to\infty}\int_\alpha^\beta \phi_n(t,x)dt=1$ holds for any α, β

such that $a \leqq \alpha < x < \beta \leqq b$. The author proves the following theorem. Let $f(x)$ be a function integrable in the sense of Denjoy-Perron, and let $\phi_n(t, x)$ be a kernel satisfying the following conditions: (1) $\phi_n(t, x) > 0$ $(n = 1, 2, \cdots)$; (2) $\int_a^b \phi_n(t, x)dt \leqq K(x) < \infty$ $(n = 1, 2, \cdots)$; (3) for fixed x, $\phi_n(t, x)$ as a function of t is increasing on the interval $[a, x]$ and decreasing in the interval $[x, b]$. Then

$$\lim_{n\to\infty} \int_a^b f(t)\phi_n(t, x)dt = f(x)$$

at every point x at which $f(t)$ is equal to the derivative of its integral. The proof is based on the following theorem. Let $g_1(x), g_2(x), \cdots$ be a sequence of functions defined in $[a, b]$, of bounded variations and such that $\lim_{n\to\infty} \int_a^c g_n(x)dx = 0$ for any $a \leqq c \leqq b$. If

$$|g_n(x)| \leqq M, \quad V_a^b(g_n) \leqq M \quad (a < x < b, n = 1, 2, \cdots),$$

then $\lim_{n\to\infty} \int_a^b f(t)g_n(t)dt = 0$ holds for any function $f(x)$ integrable in the sense of Denjoy-Perron. *M. Cotlar.*

Civin, Paul, and Chrestenson, H. E. The multiplicity of a class of perfect sets. Proc. Amer. Math. Soc. **4**, 260–263 (1953).

Soit P l'ensemble parfait obtenu en enlevant des ρ_i^{m-1} les intervalles d_i^m, ρ_1^0 étant le segment $(0, 2\pi)$, et soient

$$\epsilon_m = \sup\,(d_i^m/\rho_i^{m-1}), \quad \theta_m = \sup\,(\rho^m{}_{2i}/\rho^m{}_{2i-1}, \rho^m{}_{2i-1}/\rho^m{}_{2i}) \quad (i = 1, 2, \cdots, 2^{m-1}).$$

Si $\epsilon_m = o(1)$ et $\theta_m = O(1)$, P est un ensemble de multiplicité pour les séries trigonométriques [Bari, Uspehi Matem. Nauk (N.S.) **4**, no. 3(31), 3–68 (1949); ces Rev. **11**, 26; premier article: Fund. Math. **9**, 62–115 (1927), pp. 84–100]. Verblunsky a tenté de supprimer la condition $\theta_m = O(1)$ au moyen d'un lemme [Acta Math. **65**, 283–305 (1935), pp. 291–305]. Les auteurs montrent par un contre-exemple que ce lemme est inexact et établissent un résultat meilleur que celui de Bari: Si $\epsilon_m = o(1)$ et $\theta_m = o(1/\eta_m)$, P est un ensemble de multiplicité ($\eta_m = \sup \epsilon_n$ pour $n \geqq m$). *M. Zamansky.*

Bari, N. K. Supplement to my paper, "The uniqueness problem of the representation of a function by a trigonometric series." Uspehi Matem. Nauk (N.S.) **7**, no. 5(51), 193–196 (1952). (Russian)

The author reverts to a few problems discussed in her monograph on the uniqueness of trigonometric series [Uspehi Matem. Nauk (N.S.) **4**, no. 3(31), 3–68 (1949); these Rev. **11**, 26], corrects a statement and mentions a few new results. (1) It turns out that Verblunsky's generalization of a class of sets U (sets of uniqueness) introduced by N. K. Bary [Fund. Math. **9**, 62–115 (1927)] is invalid, not only as a proof but as a result [see also the preceding review]. (2) The old problem whether sets M (sets of multiplicity) in the ordinary sense are the same as sets M in the narrow sense has been solved, negatively, by I. I. Šapiro-Pyateckiĭ [Moskov. Gos. Univ. Učenye Zapiski **155**, Matematika **5**, 54–72 (1952); unfortunately, at this writing, the paper is not available to the reviewer]; more precisely, there exists a perfect set P of measure zero which is a set M, though for every function $F(x)$ of bounded variation and constant in the intervals contiguous to P we have $\int_0^{2\pi} e^{inx}dFx \neq o(1)$. (3) The problem of Rajchman whether every closed set U is a subset of a set H_τ [for the definition, see, e.g., the reviewer's "Trigonometrical series", Warszawa-Lwów, 1935, pp. 266, 287] has also been solved, in the negative sense, by I. I. Šapiro-Pyateckiĭ [loc. cit.].

A. Zygmund (Chicago, Ill.).

Žak, I. E. On conjugate double trigonometric series. Mat. Sbornik N.S. **31**(73), 469–484 (1952). (Russian)

Suppose $f(x, y)$ is integrable on the square $[0, 2\pi; 0, 2\pi]$ and defined periodically in the plane with period 2π in each variable. Its double Fourier series has three allied conjugate series as defined by Cesari with respect to x and y individually and jointly. It is shown that if

$$(*) \quad |f(x_1, y_1) - f(x_1, y_2) - f(x_2, y_1) + f(x_2, y_2)| \leqq K|x_2 - x_1|^\alpha |y_2 - y_1|^\beta, \quad 0 < \alpha, \beta < 1,$$

then the same is true for each of the three conjugate functions to f which may be defined as singular Cauchy integrals. The result is also proved for the subclass of those f which satisfy in addition to (*) the condition

$$(**) \qquad |f(x_1, y) - f(x_2, y)| \leqq K|x_1 - x_2|^\alpha$$

and

$$|f(x, y_1) - f(x, y_2)| \leqq K|y_1 - y_2|^\beta.$$

Cesari [Ann. Scuola Norm. Super. Pisa (2) **7**, 279–295 (1938)] showed that if f satisfies (**) for $0 < \alpha = \beta = r < 1$, then its conjugates satisfy (**) for $\alpha = \beta = r' < r$. E. M. Landis is quoted for an example of such an f two of whose conjugates do not satisfy (**) for $\alpha = \beta = r$. *G. Klein.*

Graves, Ross E. Functionals of finite degree and the convergence of their Fourier-Hermite developments. Proc. Amer. Math. Soc. **4**, 95–101 (1953).

Let $\{\alpha_k(t)\}$ be a complete orthonormal set of real functions on $[0, 1]$, let H_m denote the suitably normalized Hermite polynomial of degree m, and define

$$(*) \qquad \Psi_{m_1, \cdots, m_N}(x) = \prod_{k=1}^{N} H_{m_k}\left[\int_0^1 \alpha_k(t)dx(t)\right],$$

where x is an element of Wiener's space C of continuous functions, the integrals are in the Paley-Wiener-Zygmund sense, and $N = 1, 2, 3, \cdots, m_k = 0, 1, 2, \cdots$. The degree of the F.H. functional (*), relative to $\{\alpha_k\}$, is defined as $m_1 + \cdots + m_N$. The functionals (*) form a C.O.N. set of functionals on C. If $F \in L_2(C)$, F is said to be of degree n (relative to $\{\alpha_k\}$) if the F.H. development of F in terms of the functionals (*) contains no term of degree greater than n and at least one term of degree n. The principal results of the paper are: (1) The degree of F does not depend on the choice of the C.O.N. set $\{\alpha_k\}$. (2) The F.H. development of a functional F of finite degree converges for almost all $x \in C$. The author also proves that if $\{F_M\}$ is a sequence of functionals, all of degree $\leqq n$, which converges in the mean to a functional F, then the degree of F is $\leqq n$. *W. Rudin.*

Balagangadharan, K. A quasi-Tauberian theorem on Fourier series. J. Indian Math. Soc. (N.S.) **16** (1952), 183–190 (1953).

Let $\alpha_\mu(t) = 2^\mu \Gamma(1+\mu) t^{-\mu} J_\mu(t)$, where $J_\mu(t)$ is Bessel's function of order μ. A series $\sum a_n$ is said to be summable J_μ to s if $\sum a_n \alpha_\mu(nt)$ converges for small t and its sum tends to s as $t \to +0$ [cf. Chandrasekharan and Szász, Amer. J. Math. **70**, 709–729 (1948); these Rev. **10**, 369]. The author proves that if (i) $a_n = O(n^{-\delta})$, (ii) $s_n - s = O(n^{-\alpha})$ (C, p), where $0 < \delta < 1$, $0 < \alpha < 1$, $p > \alpha$, $\alpha p > 1 - \delta$, then $\sum a_n$ is summable J_{p-1}. He also indicates that if O is replaced by o in (ii), then (apparently) $\alpha p > 1 - \delta$ may be replaced by $\alpha p \geqq 1 - \delta$. The case $p = 1$ gives a result of Schmetterer [Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **158**, 37–59 (1950); these Rev. **12**, 329], since $\alpha_1(t) = t^{-1} \sin t$. By combining his result with one of Chandrasekharan and Szász [loc. cit.] the author obtains a corollary which includes a theorem of Loo

concerning the Cesàro continuity of the function associated with a Fourier series [Trans. Amer. Math. Soc. 56, 508–518 (1944); these Rev. 6, 126]. *L. S. Bosanquet* (London).

Chang, Han, and Rideout, V. C. A generalization of modulation spectra. Quart. Appl. Math. 11, 87–100 (1953).

The function $\sin(\omega_0 t+\sum_1^k a_r\cos\omega_r t)$ is expanded in a Fourier series. This is the problem of frequency modulation. The function $|a_1\cos\omega_1 t+a_2\cos\omega_2 t|+a_1\cos\omega_1 t+a_2\cos\omega_2 t$ is expanded in a Fourier series. This is the problem of rectification. It is assumed that the frequencies ω_i are commensurable. If the frequencies are not commensurable, it is proposed to use almost periodic function expansions.

R. J. Duffin (Pittsburgh, Pa.).

Lakshmana Rao, S. K., and Ramakrishna, B. S. A trigonometric series used in physical problems. Nature 171, 308–309 (1953).

Vilenkin, N. Ya. On the theory of Fourier integrals on topological groups. Mat. Sbornik N.S. 30(72), 233–244 (1952). (Russian)

In a previous paper [Izvestiya Akad. Nauk SSSR. Ser. Mat. 11, 363–400 (1947) (cited as V.); these Rev. 9, 224] the author generalized the theory of Fourier series to certain compact groups. This paper is an analogous generalization of the theory of Fourier integrals to locally compact separable zero-dimensional groups all of whose cyclic subgroups have compact closure. As was done in V. an ordering is introduced in the group G so that $\int_a^b f(g)dg$ (with respect to Haar measure) becomes meaningful. Using these definitions the author obtains various theorems analogous to familiar ones in the classical theory of Fourier integrals [e.g., Titchmarsh, Introduction to the theory of Fourier integrals (cited as T.), Oxford, 1937, Theorem 62]. Much interest is focused on theorems about everywhere and almost everywhere convergence of Fourier integrals; special instances of this problem (in connection with Plancherel's theorem) are reduced to the case treated in V. Finally, the paper applies the foregoing to a proof of the Plancherel theorem similar to that found in T., p. 82; and to the uniqueness proof of the Fourier transform.

G. K. Kalisch (Minneapolis, Minn.).

Vilenkin, N. Ya. On the theory of orthogonal kernels. Uspehi Matem. Nauk (N.S.) 7, no. 3(49), 63–82 (1952). (Russian)

An orthonormal kernel (o.n.k.) is a function $\varphi(x, y)$ ($a\leqq x\leqq b$ with measure μ; $A\leqq y\leqq B$ with measure ν) such that if $\nu(\Delta)<\infty$, $\omega(x;\Delta)=\int_\Delta\varphi(x, y)d\nu(y)$ is in $L_2(\mu)$ and $\int_\Delta|\varphi(x, y)|^2 d\nu(y)$ is μ-summable over every finite μ-set; and $\int_a^b\omega(x;\Delta_1)\overline{\omega(x;\Delta_2)}d\mu(x)=\nu(\Delta_1\cap\Delta_2)$. These o.n.k.'s are generalizations of systems of orthonormal functions: $\varphi_n(x)=\varphi(x, n)$. The first section deals with the occurrence of o.n.k.'s in Hellinger's approach to spectral theory in the guise of "differential solutions" [see, e.g., E. Hellinger and O. Toeplitz, Encykl. Math. Wiss., Bd. II3, Heft 9, Teubner, Leipzig, 1927, p. 1581], corresponding to the $\omega(x;\Delta)$ above. The second section deals with the connection between o.n.k.'s and Plancherel's theory of the representation of functions by integrals [Rend. Circ. Mat. Palermo 30, 289–335 (1910)] with applications to Fourier-Hankel transforms. The third section traces the analogy between o.n.k.'s and orthonormal functions by proving analogs of theorems 263, 532, 533, 585, 541, and 555 of S. Kaczmarz and H. Steinhaus [Theorie der Orthogonalreihen, Warszawa-Lwów, 1935]. The fourth section deals with some examples: If a locally compact separable group G and its character group $\hat{G}$ (with Haar measures suitably normalized) are mapped in a measure-preserving fashion onto the nonnegative reals, then the character function $g\hat{g}$ goes over into an o.n.k. $\varphi(x, y)$; the solutions of the Sturm-Liouville equation on the nonnegative reals $y''+(\lambda-q(x))y=0$ with suitable boundary conditions form an o.n.k. The fifth (last) section deals with the analog of the Gram-Schmidt orthonormalization process: given $\psi(x, y)$ find an o.n.k.

$$\varphi(x, z)=\int_0^\infty\psi(x, y)d_yL(z;y)$$

for suitable $L(x;y)$; a special case of this problem was solved by I. M. Gel'fand and B. M. Levitan [Izvestiya Akad. Nauk SSSR. Ser. Mat. 15, 309–360 (1951); these Rev. 13, 558]; some of their results are expounded and discussed from the standpoint of the theory of o.n.k.'s.

G. K. Kalisch (Minneapolis, Minn.).

Cinquini, Silvio. Funzioni quasi-periodiche ed equazioni differenziali. Univ. e Politecnico Torino. Rend. Sem. Mat. 11, 47–74 (1952).

Exposé: 1) des résultats relatifs aux fonctions presque périodiques partant de la définition de Bohr jusqu'aux extensions permises par l'intégrale de Denjoy; 2) de leurs applications: périgée et noeud lunaire, cordes et membranes vibrantes, équations différentielles. *J. Favard.*

Calderon, A. P., and Zygmund, A. On the existence of certain singular integrals. Acta Math. 88, 85–139 (1952).

In his paper on conjugate functions [Math. Z. 27, 218–244 (1927)] M. Riesz deals with Hilbert's operator

$$\tilde{f}(x)=\frac{1}{\pi}\mathrm{PV}\int_{-\infty}^{\infty}\frac{f(t)}{x-t}dt=\lim_{0<\lambda\to\infty}\tilde{f}_\lambda(x)\quad(-\infty<x<\infty)$$

where

$$\tilde{f}_\lambda(x)=\left(\int_{-\infty}^{x-1/\lambda}+\int_{x+1/\lambda}^{\infty}\right)K(x-t)f(t)dt,$$

the kernel $K(y)=(\pi y)^{-1}$. He uses complex variable methods. These are his basic results: Let $1<p<\infty$, $f(t)\in L^p(-\infty, \infty)$; then (1) $\int_{-\infty}^{\infty}|\tilde{f}_\lambda(x)|^p dx\leqq A_p{}^p\int_{-\infty}^{\infty}|f(t)|^p dt$ (that $A_2=1$ was known before; the best possible value of A_p for $p\neq 2$ is still unknown), (2) $\tilde{f}(x)=\lim\tilde{f}_\lambda(x)$ exists for almost all x, (3) $\tilde{f}_\lambda(x)\to\tilde{f}(x)$ also in the mean of order p; in (1), $\tilde{f}_\lambda(x)$ can equivalently be replaced by $\tilde{f}(x)$. The problem was subsequently solved by real variable methods, which are harder; again the difficult case $p=1$ and related ones were treated by various writers.

The authors, dedicating their work to Riesz on his 65th birthday, give a complete theory of the corresponding problem in the n-dimensional space E^n, yet in a remarkably generalised form. They had to use real variable methods. This required auxiliary results which are of great interest in themselves and for which the reader must be referred to the paper. In three chapters, they discuss the mean convergence of the singular integral, its pointwise convergence, and the application of the results to some potentials, respectively. In an addendum results on periodic functions and on linear and quadratic forms are stated, and reference is given to related recent work. Let P and Q be points in E^n, $P-Q$ be the point whose coordinates are equal to the com-

ponents of the vector QP, $|P|$ the distance of P from the origin. Let $\Omega(P)$ be defined for the points P of Σ, the surface of the unit sphere, and be such that $\int_\Sigma \Omega(P)d\sigma=0$, $|\Omega(P)-\Omega(Q)|\leqq\omega(|P-Q|)$ (P and Q on Σ), where $\omega(t)$ is increasing ($0\leqq t<\infty$), $\omega(t)\geqq t$, $\int_0^1\omega(t)t^{-1}dt<\infty$. Finally, for any P, let $K(P)=|P|^{-n}\Omega(P/|P|)$; $K_\lambda(P)=K(P)$ for $|P|\geqq 1/\lambda$, $=0$ otherwise; and

$$\bar{f}_\lambda(P)=\int_{E^n}K_\lambda(P-Q)f(Q)dQ.$$

In chapters 1 and 2 the authors, starting with the case $p=2$ which, in analogy to a familiar procedure, is dealt with by Fourier transforms in E^n, prove all the results corresponding to the above Riesz theorems on $\bar{f}_\lambda(P)$ and $\bar{f}(P)$ for $f(P)\in L^p(E^n)$ $(1<p<\infty)$; and also results on the cases $p=1$ and ∞, e.g.: if $p=1$, then $\bar{f}(P)=\lim \bar{f}_\lambda(P)$ exists almost everywhere in E^n; also $\bar{f}_\lambda(P)\to\bar{f}(P)$ in the mean of order $1-\epsilon$ ($\epsilon>0$) over any set S of finite measure, and

$$\int_S|\bar{f}_\lambda(P)|^{1-\epsilon}dP\leqq\epsilon^{-1}c|S|^\epsilon\left[\int_{E^n}|f(P)|dP\right]^{1-\epsilon}\quad(c=c(n,\Omega)),$$

which corresponds to the Kolmogoroff-Littlewood result in E^1. Other theorems concern cases like $|f(P)|\leqq M$; $\int_{E^n}|f(P)|(\log^+|f(P)|+1)dP<\infty$ and similar ones; and the singular integral $\bar{f}(P)=\int_{E^n}K(P-Q)d\mu(Q)$, where $\mu(P)$ is a completely additive function of Borel sets and is of bounded total variation over E^n.

The results are now applied to

$$u(x,y)=\int_{E^2}R^{-1}f(Q)dQ\qquad(P=(x,y),\ Q=(s,t),$$
$$R=|P-Q|=\{(x-s)^2+(y-t)^2\}^{1/2}),$$

that is the Newtonian potential of masses with density $f(s,t)$ in E^2. Theorem 1. If $f\in L(E^2)$ and $|f|\log^+|f|$ is integrable over any finite circle, then (a) over almost every line $y=$constant the integral converges and is an absolutely continuous function of x; and $\partial u/\partial x=\int\int_{E^2}R^{-3}(s-x)f(s,t)dsdt$ almost everywhere in E^2; (b) if $f\in L(E^2)$ and $|f|^q$ is integrable ($q>2$) over any finite circle, then $u(x,y)$ has a complete differential almost everywhere. In the proof of (a) it is first shown that, for $z\to 0$ ($z>0$), $\partial u(x,y,z)/\partial x-\bar{f}_z(x,y)$ converges to 0 and $\bar{f}_z(x,y)$ to a function $\bar{f}(x,y)$ in the mean of order 1 over any set of finite measure; here

$$u(x,y,z)=\int\int_{E^2}(R^2+z^2)^{-1/2}f(s,t)dsdt,$$
$$\bar{f}_z(x,y)=\int\int_{R\geqq z}R^{-3}(s-x)f(s,t)dsdt$$

(cf. Riesz's second method of proof). The theorem is shown not to admit of much improvement. Theorem 2 concerns its analogue in E^n, while Theorem 3 concerns the logarithmic potential $u(P)=\int_{E^2}\log|P-Q|^{-1}f(Q)dQ$ in E^2, and Theorem 4 the potential $\int_{E^n}|P-Q|^{-n+2}f(Q)dQ$ in E^n. *H. Kober.*

Čelidze, V. G. Integral transforms of functions of two variables. Akad. Nauk Gruzin. SSR. Trudy Mat. Inst. Razmadze **18**, 93–113 (1951). (Russian. Georgian summary)

Integrals over finite intervals being Lebesgue integrals, and integrals over a quarter-plane being limits of these, the transformations treated have the form

$$(1)\qquad F(x,y)=\int_0^\infty\int_0^\infty K(x,y,t,u)\Phi(t,u)dtdu.$$

Let $\varphi(t,u)$ be increasing over $0\leqq t,\ u<\infty$, let $\varphi(0,0)=1$, and let $\varphi(t,u)\to\infty$ as $t+u\to\infty$. Let A_φ be the class of functions $\Phi(t,u)$ which are bounded and measurable over each finite square $0\leqq t,u\leqq a$ and such that $\Phi(t,u)/\varphi(t,u)\to 0$ as $t+u\to\infty$. Let K_φ denote the class of kernels $K(x,y,t,u)$, defined over $0<x,\ y<1$, $0\leqq t,\ u<\infty$, such that $F(x,y)$ exists when $\lambda^{-1}<y/x<\lambda$ and $F(x,y)\to L$ under the condition $\lambda^{-1}<x/y<\lambda$ and $x,\ y\to 0$ whenever $\Phi(t,u)$ belongs to class A_φ and $\Phi(t,u)\to L$ as $t,u\to\infty$. The author gives four conditions on K which are said to be necessary and sufficient for K to belong to K_φ. The conditions are

$$(2)\qquad \lim_{x,y\to 0}{}^{(\lambda)}\int_0^\infty\int_0^\infty K(x,y,t,u)dtdu=1,$$

$$(3)\qquad \lim_{x,y\to 0}{}^{(\lambda)}\int_0^a dt\int_0^\infty|K(x,y,t,u)|\varphi(a,u)du=0,\quad a>0.$$

$$(4)\qquad \lim_{x,y\to 0}{}^{(\lambda)}\int_0^b du\int_0^\infty|K(x,y,t,u)|\varphi(t,b)dt=0,\quad b>0,$$

and, for some constant M,

$$(5)\qquad \int_0^\infty\int_0^\infty|K(x,y,t,u)|dtdu<M,\quad \lambda^{-1}<y/\alpha<\lambda,$$

where $\lim_{x,y\to 0}^{(\lambda)}$ means that $x,y\to 0$ under the restriction $\lambda^{-1}<y/x<\lambda$.

The long argument in support of Lemma 6, which purports to establish the necessity of the conditions (3) and (4), is incomplete at the bottom of page 98 where a formula is given and used without proof. That this formula is incorrect, and that the conditions (3) and (4) are in fact not necessary, is established by the following example. Let, when $0<x,\ y<1$,

$$(6)\qquad F(x,y)=F_1(x,y)+F_2(x,y)$$

where

$$(7)\qquad F_1(x,y)=\int_0^1\int_0^1 e^{i(x^{-1}t+y^{-1}u)}\Phi(t,u)dtdu$$

and

$$(8)\qquad F_2(x,y)=\int_{x^{-1}}^{x^{-1}+1}\int_{x^{-1}}^{x^{-1}+1}\Phi(t,u)dtdu.$$

If $\Phi(t,u)$ is Lebesgue integrable over the unit square $0\leqq t,\ u\leqq 1$, then the Riemann-Lebesgue theorem implies that $F_1(x,y)\to 0$ as $x,\ y\to 0$; and if $\Phi(t,u)$ is measurable and $\Phi(t,u)\to L$ as $u,\ v\to\infty$, then $F_2(x,y)\to L$ as $x,\ y\to 0$. Thus if $\Phi(t,u)$ belongs to the class A_φ and if $\Phi(t,u)\to L$ as $t,\ u\to\infty$, then $F(x,y)\to L$ as $x,\ y\to 0$ and hence also $F(x,y)\to 0$ as $x,\ y\to 0$ restricted by the condition $\lambda^{-1}<y/x<\lambda$. Therefore the kernel K of the transformation (6) belongs to the class K_φ. But for this kernel we have $|K(x,y,t,u)|=1$ when $0\leqq t,\ u\leqq 1$ and hence, since $\varphi(t,u)\geqq 1$,

$$(9)\qquad \int_0^1 dt\int_0^1|K(x,y,t,u)|\varphi(a,u)du\geqq 1$$

whenever $0<x,\ y<1$. Thus (3) fails to hold, and a formula similar to (9) shows that (4) fails to hold.

The paper contains three more theorems similar to that discussed above, and each of them involves the same error. It seems to the reviewer that the author's long argument that involves the errors can be modified to prove necessity of conditions such as

$$\lim_{b\to\infty}\ \limsup_{x,y\to 0}{}^{(\lambda)}\int_0^a du\int_b^\infty|K(x,y,u,v)|\varphi(u,v)dv=0,$$

but the reviewer does not assume responsibility for supplying sets of necessary and sufficient conditions to replace (2), (3), (4), and (5). *R. P. Agnew* (Ithaca, N. Y.).

Schwartz, Laurent. Transformation de Laplace des distributions. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 196–206 (1952).

This is a systematic development of the theory of Laplace transforms of distributions (in the sense of the author) in n-dimensional euclidean space along lines initiated by Bochner [Amer. J. Math. **59**, 732–738 (1937)] and others in the case of functions. Let T be a distribution on the (real) vector space X, let Ξ be the dual of X, and set $\Pi=\Xi+i\Xi$. The set of all p in Π for which $\exp(-p\cdot x)T$ is a temperate distribution is of the form $\Gamma+i\Xi$. It is shown that Γ is convex, and various differentiability, analyticity, and boundedness properties of $\exp(-p\cdot x)T$ are derived. For a given convex open set Γ, the collection Σ of all distributions T for which $\exp(-\xi\cdot x)T$ is temperate for all ξ in Γ is an algebra under convolution as product; the continuity properties of this operation are treated. The Laplace transform $F(\xi+i\eta)$ of an element of Σ is holomorphic on $\Gamma+i\Xi$, and conversely any holomorphic function on this set which for every compact subset K of Γ is bounded by a polynomial in η on $K+i\Xi$ is the Laplace transform of a unique element of Σ. If T is such that the associated convex set Γ is open, then a necessary and sufficient condition that its support be contained in the half-space $\xi\cdot x\geqq A$ (ξ a given element of Ξ) is that, for each ξ_0 in Γ and all $B<A$, $\exp(tB)F(\xi_0+t\xi+i\eta)$ be bounded by a polynomial in η, for $t\geqq 0$, where F is the Laplace transform of T (this result is ascribed to Lions, in work not yet published). The case when Γ is not open is also treated, as well as, to some extent, other spaces than that of temperate distributions. *I. E. Segal* (Chicago, Ill.).

Coon, Geraldine A., and Bernstein, Dorothy L. Some properties of the double Laplace transformation. Trans. Amer. Math. Soc. **74**, 135–176 (1953).

Unter der doppelten Laplace-Transformation versteht man die folgende Integral-Transformation

$$f(s,t)=\int_Q\int e^{-sx-ty}F(x,y)d(x,y)=L(F(x,y))$$

x, y reell, s und t komplex, Q der erste Quadrant der (x, y)-Ebene. Die Verff. geben eine Darstellung der wichtigsten Eigenschaften dieser Transformation unter möglichst allgemeinen Voraussetzungen über $F(x, y)$ und die benutzten Konvergenzbegriffe. Sie verschärfen damit bereits bekannte Resultate und beweisen auch neue Sätze.

Als Funktionen $F(x, y)$ werden in der Regel solche benutzt, die auf einem messbaren Kern von Q definiert und in jedem endlichen Rechteck R_{xy}: $0\leqq x\leqq X$, $0\leqq y\leqq Y$ im L-Sinne summierbar. Neben der gewöhnlichen und der absoluten Konvergenz des Integrales

$$f(X,Y,s,t)=\int_0^X\int_0^Y e^{-sx-ty}F(x,y)d(x,y)$$

benutzen die Verff. die beschränkte Konvergenz (to converge boundedly), wenn die folgende Bedingung erfüllt ist: $\lim_{x,y\to\infty} f(X, Y, s, t)$ konvergiert und zwar unabhängig in X und Y und es existiert eine solche Konstante M, unabhängig von X und Y, sodass

$$|f(X,Y,s,t)|\leqq M \quad \text{für alle} \quad X, Y\in Q.$$

Im 1. Teil geben die Verff. allgemeine Sätze, Lemmas und Integralformeln auf Grund klassischer Sätze aus der Theorie der reellen Funktionen betreffend die Integration obiger Funktionen $F(x, y)$. Im 2. Teil werden die wichtigsten Sätze über die Konvergenz des doppelten Laplace-Integrales formuliert, die Existenz der Konvergenzabszissen bewiesen und der Zusammenhang mit der einfachen Laplace-Transformation und ihrer iterierten Integrale besprochen. Der 3. Teil gibt die wesentlichsten Eigenschaften von $L(F_x)$, $L(F_y)$, $L(F_{xy})$, $L(F_{x,x})$ und $L(F_{x,y})$.

Im 4. Teil werden die linearen Transformationen von Q und ihr Einfluss auf die Laplace-Integrale untersucht. Im 5. Teil werden die Faltungen unter bedeutend allgemeineren Voraussetzungen als in einer frühern Untersuchung von Amerio diskutiert. Schliesslich besprechen die Verff. den Fall, dass die Laplace'sche Transformierte die Form

$$\frac{f(s,t)-f(t,t)}{s-t} \quad \text{oder} \quad \frac{f(s,t)-f(s,s)}{t-s}$$

hat und beweisen ein "Expansion theorem." *W. Saxer.*

Schwenkhagen, H. F. Ein Beitrag zur anschaulichen Deutung der Laplace-Transformation. Elektrotech. Z. **74**, 133–136 (1953).

San Juan, Ricardo. Fonctions représentables au moyen d'une intégrale de Laplace. C. R. Acad. Sci. Paris **236**, 451–452 (1953).

The author gives necessary and sufficient conditions for a function $f(s)$ to be representable as a Laplace transform $f(s)=\int_0^\infty e^{-st}F(t)dt$ convergent for $\mathrm{Re}\, s>x_0\geqq 0$. Roughly, these conditions are that $f(s)$ be holomorphic for $\mathrm{Re}\, s>x_0$, satisfying certain order restrictions there and that the complex inversion formula of the Laplace transform yield an absolutely continuous function when applied to $f(s)/s$.

I. I. Hirschman, Jr. (St. Louis, Mo.).

Gupta, H. C. Some self-reciprocal functions. Proc. Nat. Acad. Sci. India. Sect. A. **15**, 3–7 (1946).

The author proves that if a function $g(x)$ has a Mellin transform $f(s)$ which satisfies the relation

$$\Gamma(\tfrac{1}{4}r-\tfrac{1}{2}s+\tfrac{3}{4})f(s)=2^{s-1}\Gamma(\tfrac{1}{4}r+\tfrac{1}{2}s+\tfrac{1}{4})f(1-s)$$

then $g(x)$ is self-reciprocal in the Hankel transformation of order r. Several examples are given. *A. Erdélyi.*

Bose, S. K. A theorem on Whittaker transform. Bull. Calcutta Math. Soc. **44**, 51–54 (1952).

The author expresses the "Whittaker transform" of a function in terms of its Laplace transform and gives some examples. [Readers should note that the function $\mu(t)$ described as "a non-decreasing function of t" is completely determined, and that on substituting the explicit expression for $\mu(t)$ all examples turn out to be known integrals.]

A. Erdélyi (Pasadena, Calif.).

Agarwal, Ratan Prakash. Some properties of generalised Hankel transform. Bull. Calcutta Math. Soc. **43**, 153–167 (1951).

The generalized Hankel transform was introduced in a former paper [Agarwal, Ann. Soc. Sci. Bruxelles. Sér. I. **64**, 164–168 (1950); these Rev. **12**, 605]. Here the author proves certain results which hold for integral transforms of the form

$$g(x)=\int_0^\infty K(xy)f(y)dy$$

and are well known [see, for instance, Kober, Quart. J. Math., Oxford Ser. 8, 172–185 (1937)]. He applies these to self-reciprocal functions, to a combination of his transform with Laplace transforms, and to numerous examples.
A. Erdélyi (Pasadena, Calif.).

Jaiswal, J. P. On Meijer transform. II. Ann. Soc. Sci. Bruxelles. Sér. I. 66, 131–151 (1952).
This is a continuation of two earlier papers [same Ann. 66, 55–60 (1952); Math. Z. 55, 385–398 (1952); these Rev. 14, 43]. Some results on the combination of Meijer transforms with Laplace and Hankel transforms are derived and illustrated by examples. *A. Erdélyi* (Pasadena, Calif.).

Polynomials, Polynomial Approximations

Sharma, A. On the zeros of a certain polynomial. Proc. Nat. Inst. Sci. India 18, 491–493 (1952).
This paper deals with sections of the power series

$$x^{-r}[(1-x)^{-k}-1]=\sum_{n=0}^{\infty}p_n(r)x^n;\quad k>1;\quad r \text{ a positive integer.}$$

The author shows that all the zeros of $P(z)=\sum_{k=0}^{n}p_{n+k}(r)z^k$ lie in the region $\alpha\leqq|z|\leqq\beta$, where $\alpha=p_n(r)/p_{n+1}(r)$ and $\beta=p_{n+n-1}(r)/p_{n+n}(r)$. The proof consists in showing $|(\alpha-z)P(z)|>0$ for $|z|<\alpha$ and $|(1-z)g(z)|>0$ for $|z|>1$ and $g(z)=z^nP(\beta/z)$. Reviewer's note: This theorem follows at once from the author's lemma that $[p_n(r)/p_{n+1}(r)]$ is an increasing function of n and from Pólya and Szegö, "Aufgaben und Lehrsätze aus der Analysis," vol. I, p. 88, ex. 23 [Springer, Berlin, 1925]. *M. Marden.*

Venkataraman, C. S. On the zeros of a certain complex polynomial. Math. Student 18 (1950), 145–147 (1951).
Iyengar [Math. Student 6, 77–78 (1938)] showed that the section $1+z+z^2/2!+\cdots+z^n/n!$ of the power series for e^z has all its zeros in $|z|<n$. The present author generalizes this result to the section

$$f(z)=1/r!+z/(r+1)!+\cdots+z^n/(r+n)!$$

which he shows has all its zeros in the ring

$$(r+1)\leqq|z|\leqq(n+r).$$

This inequality is established by proving $|(r+1-z)f(z)|>0$ for $|z|<r+1$ and $|g(z)|>0$ for $g(z)=z^nf((n+r)/z)$ and $|z|<1$. *M. Marden* (Milwaukee, Wis.).

Dias Agudo, Fernando Roldão. On a theorem of Kakeya. Gaz. Mat., Lisboa 13, no. 53, 1–3 (1952). (Portuguese)
The theorem of Kakeya referred to here states that the moduli of all the zeros of $f(z)=p_0z^n+p_1z^{n-1}+\cdots+p_{n-1}z+p_n$ $(p_0\neq0)$ are inferior to unity if $p_0>p_1>\cdots>p_n>0$. The object of the present work is the study of cases of equality in consecutive coefficients of $f(z)$. (1) If $p_0\geqq p_1\geqq\cdots\geqq p_n>0$, the moduli of the zeros of $f(z)$ do not exceed unity. (2) If $p_0>p_1>\cdots>p_k=p_{k+1}>\cdots>p_n>0$ and $g(z)=(z-a)f(z)$, then, for $k\geqq1$,

$$\sum(1)=(|g_1|+\cdots+|g_{n+1}|)/|g_0|$$
$$=a-(1-a)(p_1+\cdots+p_{k-1}+p_{k+2}+\cdots+p_n)/p_0$$

and the moduli of the zeros of $f(z)$ are less than unity. If $k=0$, $\sum(1)=a-(1-a)(p_2+\cdots+p_n-p_0)/p_0$ and the moduli of the zeros do not attain unity when $\sum_{i=2}^{n}p_i\geqq p_0$.
(3) If

$$p_0>\cdots>p_k=p_{k+1}>\cdots>p_l=p_{l+1}>\cdots>p_m$$
$$=p_{m+1}>\cdots=\cdots>p_s=p_{s+1}>\cdots>p_n>0,$$

the moduli of the zeros of $f(z)$ are inferior to unity if $\frac{1}{2}\sum_{i=1}^{n}p_i\geqq p_k+p_l+p_m+\cdots+p_s$. (4) If $p_0=p_1=\cdots=p_n>0$, all the zeros of $f(z)$ are on the unit circle. If n is odd and $p_0=p_1\geqq p_2=p_3\geqq\cdots\geqq p_{n-1}=p_n$, then -1 is a zero of $f(z)$.
E. Frank (Chicago, Ill.).

Parodi, Maurice. Application des polynomes de Tchebicheff à la formation de matrices dont le polynome caractéristique est irréductible sur le corps des nombres rationnels. C. R. Acad. Sci. Paris 236, 567–568 (1953).
It is known that a polynomial of degree n,

$$f(x)=x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n\quad(a_n\neq0),$$

with integral coefficients is irreducible in the field of rational numbers if $n-1$ zeros are in the unit circle and one is exterior. Let A be the matrix with characteristic polynomial $f(x)$. It has been shown that $f(x)$ is irreducible if one can find a sequence of positive numbers C_k, $k=1, 2, \cdots, n$, $C_k<C_{k-1}$, such that $|a_1|>1+\sum_{j=1}^{n}|a_j|C_{n-j+1}/C_n$. Under these conditions, if $F(x)$ is a polynomial with real integral coefficients such that for $|x|<1$, $|F(x)|<1$, and $|F(x)|>1$, for $|x|>1$, the matrix $F(A)$ has an irreducible characteristic polynomial. Here the characteristic values of A are considered real. Let T_n be polynomials of Tchebicheff, where $T_n(x)-2xT_{n-1}(x)+T_{n-2}(x)=0$, $n\geqq2$, $T_0(x)=1$, $T_1(x)=x$. Let $F(x)=T_n(x)$. Then the matrices $T_n(A)$ have a characteristic polynomial irreducible in the field considered.
E. Frank (Chicago, Ill.).

Tallqvist, Hj. Die Divisibilität der Polynome. Soc. Sci. Fenn. Comment. Phys.-Math. 15, no. 6, 58 pp. (1951).
In this paper the writer obtains explicitly the condition that polynomials $f(x)$, $g(x)$ of degree m, n respectively, have a common divisor of degree r, where $1\leqq r\leqq m\leqq n\leqq6$. Explicit formulas are also obtained for the common divisors. There is also a discussion of conditions for repeated factors of a polynomial as well as of the Cardan solution of the cubic and the Ferrari and Lagrange solutions of the quartic.
L. Carlitz (Durham, N. C.).

Varnum, Edward C. Polynomial determination in a field of integers modulo p. J. Computing Systems 1, 57–70 (1953).
From a study of the expansion for integers mod 2 applied to on-off relay circuits, a generalization to any prime modulus p is made. The structure of a mod p algebra is found for the determination of a polynomial in n variables when the p^n values of the polynomial are given for all the combinations of the p values of each of the n variables.
E. Frank (Chicago, Ill.).

Carlitz, L. A note on the multiplication formulas for the Bernoulli and Euler polynomials. Proc. Amer. Math. Soc. 4, 184–188 (1953).
Consider the equation

$$(1)\qquad g_{m-1}(kx)=-\frac{2k^{m-1}}{m}\sum_{s=0}^{k-1}(-1)^sf_m\left(x+\frac{s}{k}\right),$$

where k is a fixed even integer $\geqq2$, and $f_m(x)$ is an arbitrary normalized polynomial of degree m. Then (1) determines $g_{m-1}(x)$ as a normalized polynomial of degree $m-1$. Hence (1) for a single value of k does not suffice to determine both

$f_n(x)$ and $g_{n-1}(x)$. The author proves, however, that if (1) holds for two distinct values of k, then $f_n(x)$ is identical with $B_n(x)$ except for an additive constant, and $g_{n-1}(x)$ is identical with $E_{n-1}(x)$. Here $B_n(x)$, $E_n(x)$ denote the polynomials of Bernoulli and Euler in the usual notation. Analogous results are obtained for the Bernoulli and Euler polynomials of higher order and for certain generalized Eulerian polynomials. *A. L. Whiteman* (Princeton, N. J.).

Makar, Ragy H., and Makar, Bushra H. Effectiveness at the origin of the sum set of basic sets of polynomials. Proc. Amer. Math. Soc. **4**, 246–250 (1953).

Let $\{p_n\}, \cdots, \{t_n\}$ be simple monic sets of polynomials effective at the origin; the set $\{u_n\} = \{\lambda p_n + \cdots + \gamma t_n\}$, with $\lambda + \cdots + \gamma = 1$, is called their sum. The authors prove that if $\{u_n\}$ is algebraic and each of $\{p_n\}, \cdots, \{t_n\}$ is effective at the origin, so is $\{u_n\}$. [For terminology cf. the authors, same Proc. **2**, 526–537 (1951); these Rev. **13**, 124.]

R. P. Boas, Jr. (Evanston, Ill.).

Mikhail, M. N. Derived and integral sets of basic sets of polynomials. Proc. Amer. Math. Soc. **4**, 251–259 (1953).

Let $\{p_n(z)\}$ be a basic set of polynomials; the set $\{p_n'(z)\}$ with, perhaps, one polynomial omitted, is also a basic set; and $\{1, \int_0^z p_n(x)dx\}$ is a basic set. Let D_n be the degree of the p_k of highest degree in the expansion of z^n. Let $\{p_n\}$ be effective in the annulus $a<|z|<b$, where a may be 0 and b may be ∞. The author proves the following theorems. If $\limsup D_n/n \geq 1$, the kth derived set is effective in a domain $c<|z|<b$, with $c \leq a$; the kth integral set is effective in the same domain as the original set. If $\{p_n\}$ is of order ω, the integral sets are of order at most ω; if $\limsup D_n/n \geq 1$ or if $D_n = O(n^m)$, $m \geq 1$, the derived sets are of order at most ω.

R. P. Boas, Jr. (Evanston, Ill.).

Leemans, J. Sur le développement d'une fonction par une série de polynomes donnés. Mathesis **61**, 262–269 (1952).

The author discusses convergence of the developments of a function $f(x)$ into series of polynomials of Bernoulli (which leads to a variant of the Euler-Maclaurin formula) and Legendre under the assumption that $|f^{(n)}(x)| \leq M$.

G. G. Lorentz (Toronto, Ont.).

Butzer, P. L. On two-dimensional Bernstein polynomials. Canadian J. Math. **5**, 107–113 (1953).

Let $B_{n,m}$ be the Bernstein polynomials for $f(x, y)$, with $0 \leq x \leq 1$ divided into n parts and $0 \leq y \leq 1$ divided into m parts. The author's main result is that for a bounded $f(x, y)$ any mixed partial derivative of order p of $B_{n,m}$ approaches the corresponding derivative of f at every point of the open unit square where the pth total differential of f exists, provided that n and $m \to \infty$ in such a way that n/m remains between two positive constants. If, in addition, f has continuous pth derivatives the conclusion holds uniformly without restrictions on n and m. The latter result was given by Kingsley [Proc. Amer. Math. Soc. **2**, 64–71 (1951); these Rev. **13**, 128] with a different proof.

R. P. Boas, Jr. (Evanston, Ill.).

Butzer, P. L. Dominated convergence of Kantorovitch polynomials in the space L^p. Trans. Roy. Soc. Canada. Sect. III. (3) **46**, 23–27 (1952).

The Kantorovitch polynomials corresponding to $f(x)$ are

$$P_n(x) = \sum_{\nu=0}^{n} (n+1)\binom{n}{\nu} x^\nu (1-x)^{n-\nu} \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} f(t)dt.$$

By writing this expression in the form of a singular integral the author is led to a short proof of a theorem of Lorentz [forthcoming book] that $P_n(x) \to f(x)$ almost everywhere when $f(x) \in L$. His main result is that $\{P_n(x)\}$ converges dominatedly to $f(x)$ when $f(x) \in L^p$, $p>1$, whence $P_n(x) \to f(x)$ in the L^p metric. The corollary was obtained otherwise by Lorentz and by E. Levi [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **9**, 242–246 (1950); **10**, 360–364 (1951); these Rev. **12**, 701; **13**, 343].

R. P. Boas, Jr. (Evanston, Ill.).

Popov, B. S. On a property of the derivatives of orthogonal polynomials. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **4**, no. 5, 8 pp. (1951). (Macedonian and English)

Let $R_n(x) = c_n d^n\{(x-a)^n(x-b)^n\}/dx^n$, denote by $R_n^{(r)}(x)$ its rth derivative. Using Leibnitz' rule for the derivative of a product, it is shown that

$$(1) \qquad R_n^{(r)}(a) = c_n \binom{n+r}{r} n! \binom{n}{r} r!(a-b)^{n-r}.$$

This generalizes the well-known corresponding result for Legendre's polynomials $P_n(x)$ [see du Plessis, Proc. Amer. Math. Soc. **2**, 950 (1951); these Rev. **13**, 553; also Grosswald, ibid. **1**, 553–554 (1950); these Rev. **12**, 178]. Using hypergeometric functions, the author gives an alternative proof of (1) for Legendre's polynomials, identical to an (unpublished) proof of Rainville, which the author does not seem to know. Gegenbauer's polynomials, defined by the generating function (2) $(1-2xz+z^2)^{-\nu} = \sum_n C_n^\nu(x) z^n$ have derivatives satisfying

$$(3) \qquad \left[C_n(x)\right]_{x=1}^{(r)} = 2^r \Gamma(n+2\nu+r)\Gamma(\nu+r)/\Gamma(n+1-r)\Gamma(2\nu+2r)\Gamma(\nu).$$

(3) is proved by differentiating (2) (method identical to that of du Plessis, loc. cit.; observe that $C_n^{\frac{1}{2}}(x) = P_n(x)$) and, alternatively, by hypergeometric functions. Similarly it is mentioned that the derivatives of Tchebycheff's polynomials

$$T_n(x) = \frac{(-1)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} (1-x^2)^{\frac{1}{2}} \frac{d^n}{dx^n}(1-x^2)^{n-\frac{1}{2}}$$

satisfy $T_n^{(r)}(1) = n \cdot 2 \cdot 4 \cdots (2r-2)\binom{n+r-1}{2r-1}$. *E. Grosswald.*

Special Functions

Bagchi, Hari das, and Chatterji, Phatik chand. Note on certain functional equations, connected with Hermite and Weber functions. Ann. Scuola Norm. Super. Pisa (3) **6**, 75–83 (1952).

In two former papers [same Ann. (3) **5**, 71–80 (1951); J. Roy. Asiatic Soc. Bengal. Sci. **16**, 17–26 (1950); these Rev. **13**, 554, 649] the authors made some remarks on the differential equations and recurrence and differentiation formulas satisfied by $D_n(z)$. They now repeat more or less the same remarks for

$$\exp\left[-\frac{1}{4}\left(2k - \frac{\lambda^2}{\mu}\right)z^2\right] D_n\left(\frac{\lambda z}{\sqrt{\mu}}\right)$$

both for general k, λ, μ and some special values of these parameters. *A. Erdélyi* (Pasadena, Calif.).

Giuliano, Landolino. Su alcune relazioni integrali fra funzioni di Bessel di prima e di seconda specie. Ann. Scuola Norm. Super. Pisa (3) **6**, 17–30 (1952).

The author evaluates some integrals of Lommel's type [G. N. Watson, Bessel functions, 2nd ed., Cambridge, 1944, §5.11, 5.12; these Rev. **6**, 64] in which one of the limits is ∞. *A. Erdélyi* (Pasadena, Calif.).

Charnes, A. Note on the zeros of modified Bessel function derivatives. Bull. Calcutta Math. Soc. **43**, 133–134 (1951).

The author obtains information on the location and number of zeros of the function $K_\nu'(z)$. The notation is according to the treatise of Watson. *G. Szegö* (Stanford, Calif.).

Cooke, J. C. Some properties of Legendre functions. Proc. Cambridge Philos. Soc. **49**, 162–164 (1953).

Correcting a statement of Nicholson, the author proves the formula

$$(n-m)(n+m+1)\int_{-1}^{+1} Q_n(x)Q_m(x)dx = -2\left(\frac{1}{m+1}+\frac{1}{m+2}+\cdots+\frac{1}{n}\right)$$

where $n>m$ and $n-m$ is even. In case $n-m$ is odd the integral is obviously zero. Suggested by a result of Nicholson, a formula is proved involving Bessel functions and associated Legendre functions. *G. Szegö.*

Villari, Gaetano. Sugli estremi relativi dei polinomi di Legendre. Boll. Un. Mat. Ital. (3) **7**, 421–423 (1952).

Let $\mu_{r,n}$ be the rth maximum of $|P_n(x)|$ in decreasing order, $1>\mu_{1,n}>\mu_{2,n}>\cdots>\mu_{k,n}$, $k=[n/2]$. It was shown by the reviewer [same Boll. (3) **5**, 120–121 (1952); these Rev. **12**, 332] that $\mu_{r,n}$ is decreasing as n increases, r fixed. It is shown that $\lim_{n\to\infty}\mu_{r,n}=h_r$ is positive. Bounds for h_1 and h_2 are given. The result (not the method) is clear since it is well known that h_r is the rth maximum of $|J_0(x)|$. The inequalities of the note follow easily from the familiar tables. *G. Szegö* (Stanford, Calif.).

Grosswald, Emil. On sums involving binomial coefficients. Amer. Math. Monthly **60**, 179–181 (1953).

Carlitz, L. Note on a formula of Grosswald. Amer. Math. Monthly **60**, 181 (1953).

In the first of these two papers Grosswald uses properties of Legendre polynomials and of hypergeometric functions to sum three series of products of binomial coefficients. In the second note Carlitz points out that these sums are known [cf. Whittaker and Watson, Modern analysis, 4th ed., Cambridge, 1927, p. 298, Ex. 13]. *A. Erdélyi.*

Carpani, Ada. Sopra un nuovo sviluppo asintotico per la funzione ipergeometrica confluente Ψ di Tricomi. Univ. e Politecnico Torino. Rend. Sem. Mat. **11**, 261–269 (1952).

Hadamard has shown that asymptotic expansions of Bessel functions may be modified so as to become convergent infinite series of incomplete gamma functions together with a negligible remainder. In the present paper the corresponding work is carried out for confluent hypergeometric functions, and the results are illustrated by numerical computations. *A. Erdélyi* (Pasadena, Calif.).

Bose, N. N. On MacRobert's E-function. Bull. Calcutta Math. Soc. **44**, 63–68 (1952).

The particular E-functions occurring here are confluent hypergeometric functions, and the formulas are transcriptions in the alternative notation of known integral formulas involving confluent hypergeometric functions. *A. Erdélyi* (Pasadena, Calif.).

Biedenharn, L. C. An identity satisfied by the Racah coefficients. J. Math. Physics **31**, 287–293 (1953).

The Racah coefficients are

$$W(abcd;ef) = C\cdot {}_4F_3\left[\begin{matrix} e-a-b, e-c-d, f-a-c, f-b-d \\ -a-b-c-d-1, e+f-a-d, e+f-b-c \end{matrix};1\right]$$

where C^2 is a product of factorials and reciprocal factorials. The author obtains a number of identities which facilitate the tabulation of these coefficients. These coefficients occur as transformation coefficients from one coupling of angular momentum vectors to another coupling and the proofs are based on this interpretation. *A. Erdélyi.*

Carlitz, L. Note on a formula of Szily. Scripta Math. **18** (1952), 249–253 (1953).

Using a formula of Kummer in the theory of hypergeometric series [cf. W. N. Bailey, Generalized hypergeometric series, Cambridge, 1935, p. 9] the author evaluates the sum of the series $\sum_{r+s=k}(-1)^r\binom{a}{r}\binom{b}{s}$ for arbitrary a, b (not necessarily integral values). His result includes as a special case the following formula due to K. v. Szily [Math. Nat. Ber. Ungarn **12**, 84–91 (1895)]:

$$\sum_{r=0}^{b}(-1)^r\binom{2a}{a-r}\binom{2b}{b-r}=\binom{2a}{a}\binom{2b}{b}\Big/\binom{a+b}{a}, \quad a\geqq b.$$

The author also treats in a similar manner some further extensions of Szily's sum. *A. L. Whiteman.*

Meijer, C. S. Expansion theorems for the G-function. II. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 483–487 (1952).

[For part I see same Proc. **55**, 369–379 (1952); these Rev. **14**, 469.] In this part, theorem 2 gives conditions under which the expansion

$$G_{pq}^{mn}\left(\lambda\omega\middle|\begin{matrix}a_r\\ b_s\end{matrix}\right)=\sum_{k=0}^{\infty}\frac{1}{k!}\left(1-\frac{1}{\lambda}\right)^k G_{p+1,q+1}^{m,n+1}\left(\omega\middle|\begin{matrix}1-k, a_r\\ b_s, 1\end{matrix}\right)$$

is valid. The author also establishes the connection between his present expansions and the multiplication theorems which he proved in an earlier paper [ibid. **44**, 1062–1070 (1941); these Rev. **8**, 155]. *A. Erdélyi.*

***Gonzalez Dominguez, Alberto. Distributions and analytic functions.** Symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Diciembre, 1951, pp. 91–106. Centro de Cooperación Científica de la Unesco para América Latina, Montevideo, Uruguay, 1952. (Spanish)

The author gives a rigorous treatment of a number of singular functions employed in quantum electrodynamics, via the Schwartz distribution theory. Some of the formulas of Schwinger, Feynman and Dyson are obtained more simply in their present rigorous formulation. *I. E. Segal* (Chicago, Ill.).

Sevdić, Milenko. Définition des fonctions hyperboliques au moyen du théorème de binome. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **7**, 140–168 (1952). (Serbo-Croatian. French summary)

For a positive integer n, the author sets

$$(x\pm y)^n = X(n)\pm Y(n),$$

develops algebraic relationships satisfied by X and Y, uses these to complete the definitions of $X(n)$, $Y(n)$ for all real n, and discusses the relation with hyperbolic functions.

A. Erdélyi (Pasadena, Calif.).

***Ryžik, I. M., i Gradšteĭn, I. S. Tablicy integralov, summ, ryadov i proizvedeniĭ. [Tables of integrals, sums, series and products.]** 3d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 464 pp. 20.45 rubles.

Table of contents: Introduction; Elementary functions; Indefinite integrals; Definite integrals of elementary functions; Definite integrals of special functions; Integral transforms; Special functions; Numerical tables. The material is mostly from standard sources with the sources indicated; e.g., Bierens de Haan, Nouvelles tables d'intégrales définies [Amsterdam, 1867], Magnus and Oberhettinger, Formeln und Sätze . . . [Springer, Berlin, 1948; these Rev. **10**, 38], Whittaker and Watson, A course of modern analysis [Cambridge, 1927].

Kopineck, Hermann-Josef. Zweizentrenwechselwirkungsintegrale. III. Integrale mit $2p$- und wasserstoffähnlichen $2s$-Funktionen. Z. Naturforschung **7a**, 785–800 (1952).

Continuation of two earlier papers [same Z. **5a**, 420–431 (1950); **6a**, 177–183 (1951); these Rev. **12**, 410; **13**, 130] with 9 pages of formulas, 3 pages of numerical values, and corrections to the earlier papers. *A. Erdélyi.*

Brennan, R. O., and Mulligan, J. F. Two-center heteronuclear hybrid Coulomb-exchange integrals. J. Chem. Phys. **20**, 1635–1644 (1952).

The integrals discussed in this paper measure the interaction of one electron exchanged between two atoms a and b, and the charge distribution $\bar{\chi}_a\bar{\chi}'_b$ of the second electron on atom b. The atomic orbitals are Slater's $2s$ and $2b$ orbitals, χ, χ', $\bar{\chi}$, $\bar{\chi}'$ are four different real orbitals, 1 and 2 number the electrons, and the integrals are of the form

$$[\chi_a\chi'_b|\bar{\chi}_b\bar{\chi}_b'] = \int\int \chi_a(1)\bar{\chi}_b(2)\frac{2}{r_{12}}\chi'_b(1)\bar{\chi}_b'(2)dv_1dv_2.$$

The integration over the coordinates of electron 2 may be carried out easily in polar coordinates. For the subsequent integration over the coordinates of electron 1 the authors use dipolar coordinates. They express 30 integrals in terms of certain standard integrals (incomplete gamma functions, confluent hypergeometric functions, and the like) and give recurrence and reduction formulas which enable them to compute all the required integrals from tables of elementary functions and the exponential integral function.

A. Erdélyi (Pasadena, Calif.).

Murai, Tomokazu, and Araki, Gentaro. Calculation of heteronuclear molecular integrals. Progress Theoret. Physics **8**, 615–638 (1952).

This paper contains reduction formulas and numerical tables for certain two-center integrals occurring in nuclear physics [see also Barnett and Coulson, Philos. Trans. Roy. Soc. London. Ser. A. **243**, 221–249 (1951); these Rev. **12**, 702; Roothaan, J. Chem. Phys. **19**, 1445–1458 (1951); these Rev. **13**, 843; Rüdenberg, ibid. **19**, 1459–1477 (1951); these Rev. **13**, 843; the paper reviewed above, and other references given in these papers]. The integrals are expressed in terms of the following standard integrals

$$A_n(p) = \int_1^\infty x^n e^{-px}dx, \quad B_n(q) = \int_{-1}^1 x^n e^{-qx}dx$$

$$F_{mn}(p,q) = \int_1^\infty dx\int_{-1}^1 dy(x+y)^{-m}(-y)^n(1-y^2)^{m-1}e^{-px-qy}$$

$$J_n(p,q) = \int_1^\infty dx\int_{-1}^1 dy(x^2-1)(1-y^2)(x^2-y^2)(x+y)^n e^{-px-qy}$$

$$K_n(p,q) = \int_1^\infty dx\int_{-1}^1 dy(x^2-1)(1-y^2)(x^2-y^2)(1+xy)^2 \times(x+y)^{n-2}e^{-px-qy}.$$

Of these, J and K may be expressed in terms of A, B, F; and F may be expressed recurrently in terms of A, B and the exponential integral function. The authors give numerical tables for J_n and K_n, $n=-3(1)2$, and for F_{mn}, $0\leqq m\leqq 4$, $0\leqq n\leqq 9$, in each case for assorted values of p, q. They also tabulate certain nuclear integrals. A 35D table of $\log_e n$ for $n=2(1)20$ is appended. *A. Erdélyi.*

Knudsen, H. Lottrup. On the calculation of some definite integrals in antenna theory. Appl. Sci. Research B. **3**, 51–68 (1952).

The mutual radiation resistance of a pair of identical antennas may be computed in two different ways, and a comparison of the results leads to the expression of certain definite integrals in terms of tabulated functions. The four most interesting integrals obtained thus by the author are

$$\int_0^{\pi/2} f(a,\theta)g(b,\theta)d\theta$$

where

$$f(a,\theta) = \frac{[\cos(a\cos\theta)-\cos a]^2}{\sin\theta} \quad \text{or} \quad \frac{\sin^3\theta\sin^2(a\cos\theta)}{\cos^2\theta}$$

$$g(b,\theta) = J_0(b\sin\theta) \quad \text{or} \quad \cos(b\cos\theta).$$

Three of these are expressed in terms of the sine and cosine integral, and the fourth in terms of the generalized sine integral. *A. Erdélyi* (Pasadena, Calif.).

Lewin, L. Theory of wave-guide-fed slots radiating into parallel plate regions. J. Appl. Phys. **24**, 232 (1953).

The author shows that

$$Q(\xi) = \int_0^{\frac{1}{2}\pi}\frac{\cos^2(\frac{1}{2}\pi[1-\xi^2]^{1/2}\sin\theta)}{1-(1-\xi^2)\sin^2\theta}d\theta = \frac{\pi}{4}\int_0^\pi \sin x J_0(x[1-\xi^2]^{1/2})dx$$

and that $Q(0)=\pi^2J_1(\pi)/4$. The function occurs in the subject of the title.

Harmonic Functions, Potential Theory

Ninomiya, Nobuyuki. Sur le caractère fonctionelle de la solution du problème de Dirichlet. Math. J. Okayama Univ. **2**, 41–48 (1952).

Let F denote the frontier of a domain D in R^n ($n\geqq 3$), and let $\mathfrak{F}$ denote the set of all real continuous functions defined

on F. Then it is well known that the generalized (Wiener) solution $H_f^D(M)$ of the Dirichlet problem for D and f is a non-negative linear functional on $\mathfrak{F}$ for each fixed $M \in D$. In this paper, the author considers the converse question. Given a non-negative linear functional $A_M(f)$, on $\mathfrak{F}$, for $M \in D$; when is $A_M(f)$ a (generalized) solution to the Dirichlet problem for D and f? The author uses the methods of H. Cartan [Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **22**, 221–280 (1946); these Rev. **8**, 581] to improve upon results due to Keldych [C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 308–309 (1941); these Rev. **6**, 155] and Inoue [Mem. Fac. Sci. Kyūshū Univ. A. **5**, 69–84 (1950); these Rev. **12**, 609].

For example, let $A_M(f)$ be a non-negative linear functional defined on $\mathfrak{F}$, for $M \in D$, such that (a) for each $f_\lambda \in \mathfrak{F}$ given by a spherical distribution λ, one has

$$U^\lambda(M) = \int (1/MQ^{n-2})d\lambda(Q) \geqq A_M(f_\lambda),$$

and (b) for each $f \in \mathfrak{F}$ given by a continuous potential of a positive distribution σ on the complement of D, one has $U^\sigma(M) = A_M(f) = A_M(U^\sigma)$. Then $A_M(f) = H_M^D(f)$ for all $f \in \mathfrak{F}$. For, one can write $A_M(f) = \int_F f(Q)d\nu_M(Q)$, where $\nu_M(Q) = 1/MQ^{n-2}$. From (b) above we have, if $U^\mu(Q)$ is the continuous potential of a positive distribution μ on the complement of D, then

$$\int_F U^{\nu_M} d\mu(Q) = \int_F U^\mu(Q)d\nu_M(Q) = A_M(U^\mu)$$
$$= U^\mu(M) = \int U^{\epsilon_M}(Q)d\mu(Q),$$

where ϵ_M denotes the distribution with unit mass at M. Hence $U^{\nu_M}(Q) = U^{\epsilon_M}(Q)$ for all points of the complement of D (except for a set of capacity zero) and (by (a)) it follows that ν_M is obtained by "sweeping ϵ_M out" of D onto F. Let $\alpha = \sum_{i=1}^{\infty} a_i\lambda_i$ be a positive distribution obtained from spherical distributions λ_i with rational radii and rational centers, and from positive numbers a_i, such that $U^\alpha(Q)$ is continuous in R^n. Let $A_M(f)$ be a non-negative linear functional defined on $\mathfrak{F}$ for $M \in D$ such that (b) holds, and (c) for $f \in \mathfrak{F}$ given by $U^\alpha(Q)$, one has $A_M(f) = H_M^D(U^\alpha)$. Then $A_M(f) = H_M^D(f)$ for all $f \in \mathfrak{F}$.

The author concludes with a strengthening of Keldych's original result.

M. Reade (Ann Arbor, Mich.).

Tsuji, Masatsugu. Fundamental theorems in potential theory. J. Math. Soc. Japan **4**, 70–95 (1952).

The author gives short, neat, and revealing proofs of most of the fundamental theorems of potential theory; he covers the maximum principle, capacity, Green's functions, the Dirichlet problem, elliptic capacity, and two applications to function theory. The author confines himself to the logarithmic potential so that he may apply the results to function theory. For example, the author proves that if D is a domain in the w-plane, with boundary Γ of positive capacity, if E is a closed subset of Γ such that E and $\Gamma - E$ are at a positive distance, and if D is mapped conformally onto $|z| < 1$, then the image of E on $|z| = 1$ has positive measure if and only if E has positive capacity. [The reviewer believes that the proofs of the uniqueness theorems (9, 12) are incomplete.]

M. Reade (Ann Arbor, Mich.).

***Lokki, Olli. Über harmonische Funktionen mit endlichem Dirichletintegral.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 239–242. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

Careful discussion of an extremal problem solved by the kernel function.

P. R. Garabedian (Stanford, Calif.).

Plūme, Z. Solution of Neumann's boundary problem in potential theory by singular integral equations. Latvijas PSR Zinātņu Akad. Vēstis **1950**, no. 8(37), 119–126 (1950). (Latvian. Russian summary)

The interior Neumann problem for a plane domain bounded by a curve L having a Hölder continuously turning tangent (but in general no radius of curvature) is solved by using singular integral equations. The solution is represented as a potential of a double layer. [For a similar treatment of the Dirichlet problem see G. Bertrand, Bull. Sci. Math. (2) **47**, 282–288, 298–307 (1923); N. Musheliśvili, Soobšč. Gruzin. Filial. Akad. Nauk SSSR **1**, 99–106, 169–170 (1940); these Rev. **1**, 314.]

L. Bers (New York, N. Y.).

Pavinskiĭ, P. P. The electrostatic problem for the sphere. Leningrad. Gosudarstv. Univ. Učenye Zapiski **120**, Ser. Fiz. Nauk **7**, 134–155 (1949). (Russian)

The mathematical problem treated here is as follows. Let r, ϑ, φ be polar coordinates, $u(r, \vartheta, \varphi)$ a given function harmonic for $r < 1$ and regular also for $r = 1$ except for isolated points or curves. One has to find two harmonic functions $u_i(r, \vartheta, \varphi)$, $u_e(r, \vartheta, \varphi)$ defined for $r < 1$, $r > 1$ respectively, such that u_e vanishes at infinity of order at least $1/2$, so that the boundary conditions

$$u_i = u_e, \quad \alpha \partial U_i/\partial r + (1-\alpha)\partial U_e/\partial r = u$$

should hold on $r = 1$. Here α is a fixed number, $0 < \alpha < 1$. In the physical problem u is given in terms of another harmonic function $\Phi(r, \vartheta, \varphi)$ by the relation $u = (1-2\alpha)r(\partial\Phi/\partial r)$. The author remarks that expansions in spherical harmonics cannot be used if u is permitted to have irregularities on $r = 1$. He gives explicit integral formulas for u_i and u_e in terms of either u and Φ. These formulas are used for the detailed investigation of the case where Φ is the potential of a point course.

L. Bers (New York, N. Y.).

Svešnikov, A. G. The principle of limiting absorption for the metaharmonic equation. Doklady Akad. Nauk SSSR (N.S.) **86**, 231–234 (1952). (Russian)

It is shown that the exterior boundary-value problem for the metaharmonic equation

$$\Delta^n u + a_1\Delta^{n-1}u + \cdots + a_n u = -f(M)$$

has a unique solution bounded at infinity provided that in the characteristic equation

$$x^n + a_1x^{n-1} + \cdots + a_n \equiv (x-\alpha_1^2)\cdots(x-\alpha_n^2) = 0$$

the α_k are all of the form $\alpha_k = \xi_k + i\eta_k$ ($\xi_k \neq 0$); that is, provided the characteristic equation has no negative real roots. If we let $\xi_{k_0} \to 0$, so that the metaharmonic equation tends to an equation whose characteristic equation has a negative real root, then the solution tends to a solution of the limiting equation.

E. F. Beckenbach (Los Angeles, Calif.).

Differential Equations

Besicovitch, A. S. On the existence theorem for the differential equation $dy/dx=\phi(x, y)$. J. London Math. Soc. **28**, 110–112 (1953).

The differential equation $y'=f(x, y)$ is known to have a solution if f is measurable with respect to x for every fixed y and continuous with respect to y. This paper shows that measurability cannot replace continuity in the condition by constructing an f for which the equation has no solution.

J. M. Thomas (Durham, N. C.).

Markus, L. A uniqueness theorem for ordinary differential equations involving smooth functions. Proc. Amer. Math. Soc. **4**, 88 (1953).

The uniqueness of the solution of $y'=f(x, y)$, $y(0)=y_0$, where f is continuous and satisfies

$$|f(x, y+h)+f(x, y-h)-2f(x, y)| \leqq kh$$

for all small positive h and constant positive k, is proved by applying Osgood's theorem. *J. M. Thomas.*

Zwirner, Giuseppe. Sopra il problema di Nicoletti per le equazioni differenziali ordinarie d'ordine n. Ann. Univ. Ferrara. Sez. VII. (N.S.) **1**, 1–7 (1952).

For the problem $y^{(n)}=f(x, y, y', \cdots, y^{(n-1)})$, $y(x_k)=c_k$ $(k=1, \cdots, n)$, an existence theorem is given that generalizes previous theorems of the author [Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 217–222 (1942); these Rev. **8**, 206] and S. Cinquini [Ann. Scuola Norm. Super. Pisa (2) **9**, 61–77 (1940); these Rev. **2**, 51]. *F. A. Ficken.*

Germay, R. H. Sur l'intégration des systèmes d'équations récurro-différentielles par la méthode des approximations successives. Bull. Soc. Roy. Sci. Liège **21**, 309–313 (1952).

The author considers the infinite system of differential equations

$$dy_n^{(j)}/dx=F_n^{(j)}(x, y_n^{(1)}, \cdots, y_n^{(p)}, \cdots, y_{n+\varepsilon}^{(1)}, \cdots, y_{n+\varepsilon}^{(p)}),$$
$$(j=1, \cdots, p;\ n=1, 2, 3, \cdots).$$

It is supposed that each of the functions $F_n^{(j)}$ is defined as the limit of an infinite sequence of known functions. Assuming suitable continuity conditions of the usual kind, the author constructs a solution by a method of successive approximations which makes use of the defining sequences, but not of the functions $F_n^{(j)}$ themselves.

L. A. MacColl (New York, N. Y.).

Gates, Leslie D., Jr. Differential equations in the distributions of Schwartz. Iowa State Coll. J. Sci. **27**, 105–111 (1952).

L'auteur étudie les équations différentielles

$$L(T)=\sum_i p_i(x)T^{(n-i)}=S,$$

p_i fonctions indéfiniment différentiables, S distribution. Il donne une expression de la solution quand le transposé ${}^tL$ applique $(\mathfrak{D})$ sur un sous-espace fermé de codimension finie. Il applique ensuite les formules de convolution avec la solution élémentaire à des problèmes d'équation de la chaleur. *L. Schwartz* (Paris).

Caligo, Domenico. Sopra una classe di equazioni differenziali non lineari. Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) **1**, 24 pp. (1952)=Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **342** (1952).

The author considers the differential equation

$$\Phi(y'', y', y)=0,$$

where Φ is a homogeneous quadratic polynomial with constant coefficients. Two methods are given for obtaining the general solution, one of which can be described as follows. If y, y', y'' are regarded as homogeneous coordinates of a point in a plane, the differential equation represents a conic. A parametric representation of the conic is obtained, and it results that the parameter is determined as a function of the independent variable by a differential equation of the first order with separable variables. A quadrature completes the solution. A detailed study is made of the algebraic aspects of the method, and the various possibilities are illustrated by examples. The given differential equation possesses solutions of the form $y=A\exp(\alpha x)$, where A is an arbitrary constant and α is a root of an auxiliary algebraic equation. An investigation shows that in some cases these solutions belong to the general solution, and in other cases they are singular solutions.

It is pointed out that the methods used are applicable also in the case of an equation of the form $\Phi(y''', y'', y', y)=0$, where Φ is a homogeneous polynomial with constant coefficients. However, no details concerning such extensions are given. *L. A. MacColl* (New York, N. Y.).

Evans, Robert L. Solution of linear ordinary differential equations containing a parameter. Proc. Amer. Math. Soc. **4**, 92–94 (1953).

The author obtains the power series

$$(1)\qquad y(\lambda, x)=\sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty} b_{\alpha,\beta}\lambda^{p\beta-\alpha}x^{\beta}$$

as the solution of the initial value problem in which

$$(2)\qquad y^{(n)}-\sum_{j=1}^{n}\lambda^{pj}P_j(\lambda, x)\cdot y^{(n-j)}=0,$$

$$(3)\qquad y^{(j)}(\lambda, 0)=(j!)\sum_{\alpha=0}^{\infty} b_{\alpha,j}\cdot\lambda^{-\alpha} \quad \text{for } j=0, 1, \cdots, n-1,$$

where

$$(4)\qquad P_j(\lambda, x)=\sum_{\mu=-pm_j}^{\infty}\ \sum_{\nu=\lfloor(p-\mu-1)/p\rfloor}^{\infty} a_{j,\mu,\nu}\lambda^{-\mu}x^{\nu},$$

n, p, and m_j are all integers such that n, $p>0$, $m_j\geqq 0$, and the series in (3) and (4) converge for $|x|<x_0$ and $|\lambda|>\lambda_0$. The solution (1), which is obtained by making the transformations $z=\lambda^p x$, $w(\lambda, z)=y(\lambda, z/\lambda^p)$ and solving the corresponding problem for $w(\lambda, z)$, also converges for $|x|<x_0$ and $|\lambda|>\lambda_0$. A recursion formula for those $b_{\alpha\beta}$ in (1) which do not appear in (3) is given. The author points out the connection of his result with those of other authors who have treated special cases of equation (2). [See, e.g., Langer, Trans. Amer. Math. Soc. **67**, 461–490 (1949); these Rev. **11**, 438; Cherry, ibid. **68**, 224–257 (1950); these Rev. **11**, 596.] *C. E. Langenhop* (Ames, Iowa).

Gradšteĭn, I. S. Linear equations with variable coefficients and small parameters in the highest derivatives. Amer. Math. Soc. Translation no. **82**, 32 pp. (1953).

Translated from Mat. Sbornik N.S. **27(69)**, 47–68 (1950); these Rev. **12**, 260.

Hartman, Philip, and Wintner, Aurel. On the behavior of the solutions of real binary differential systems at singular points. Amer. J. Math. **75**, 117–126 (1953).

Let Γ: $x=x(t)$, $y=y(t)$ be a solution path of $\dot{x}=f(x, y)$, $\dot{y}=g(x, y)$ tending to the critical point

$$x=y=0 \quad (f(0, 0)=g(0, 0)=0),$$

and let $\theta(t)=\tan^{-1}y(t)/x(t)$, $\psi(t)=\tan^{-1}\dot{y}(t)/\dot{x}(t)$. The questions whether (1) $\lim\theta(t)=\theta_0$ exists, (2) $\lim\psi(t)=\theta_0$ (mod π), are investigated. Assumptions on f, g, apart from continuity, are: (a) there is a periodic dense open set A such that $\lim_{r\to\infty}r^{-1}f(r\cos\theta, r\sin\theta)=F(\theta)$, $\lim_{r\to\infty}r^{-1}g(r\cos\theta, r\sin\theta)=G(\theta)$ exist uniformly in any compact subset of A, and $F^2(\theta)+G^2(\theta)>0$ in A; (b) the subset A_0 of A at which $G(\theta)\cos\theta-F(\theta)\sin\theta=J(\theta)$ vanishes has no cluster points in A. It is shown that: (i) If (1) holds, then θ_0 is either in A_0 or not in A; (ii) if (1) does not hold, then $|\theta(t)|\to\infty$; (iii) if $J(\theta)$ changes sign at $\theta=\theta_0$ in A_0, then at least one solution path Γ satisfies (1); (iv) if $J(\theta)$ takes both positive and negative values, then (1) holds (for some θ_0) for every solution path Γ. These results are applied to the special case

$$f(x, y)=\alpha x+\beta y+o(r),\quad g(x, y)=\gamma x+\delta y+o(r).$$

G. E. H. Reuter (Manchester).

Markus, L. Escape times for ordinary differential equations. Univ. e Politecnico Torino. Rend. Sem. Mat. **11**, 271–277 (1952).

The author is concerned with finding conditions on real ordinary differential equations which have solutions with infinite escape times. Denote the positive and negative escape times, with respect to an open set $R\subset E^n$ and the solution through the point x, by $T_+(x)$ and $T_-(x)$, respectively. The following theorems are among those proved after certain continuity properties, of interest in themselves, are obtained for the escape times. Let $\dot{x}_i=f_i(x)$, $i=1, 2, \cdots, n$, $f_i\in C^{(1)}$, and $R=E^n$. If

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left|f_i\frac{\partial f_i}{\partial x_j}f_j\right|=O\left(\sum_{i=1}^{n}f_i^2\right)$$

as $|x|=\max(|x_1|, \cdots, |x_n|)\to\infty$, then the escape times $T_+(x)=+\infty$ and $T_-(x)=-\infty$. Let $A(x_1)$, $B(x_1)\in C^{(1)}$ with $A(x_1)<0$ and $B(x_1)=O(x_1^{2n+1}A(x_1))$ as $|x_1|\to\infty$. Then the differential systems

$$\dot{x}_1=x_2,\quad \dot{x}_2=A(x_1)x_2^{2n+1}+B(x_1),\quad n=0, 1, 2, \cdots, R=E^2,$$

all have $T_+(x)=\infty$. *C. J. Titus* (Ann Arbor, Mich.).

Ascoli, Guido. Questioni asintotiche nel campo delle equazioni differenziali non lineari. Rend. Sem. Mat. Fis. Milano **22** (1951), 63–73 (1952).

An expository article concerning the asymptotic properties, as $t\to\infty$, of certain differential equations of the form $\dot{x}=Ax+B(x, t)$ (vectorial notation). It includes an account of the author's own researches (partly published, partly in press) on the differential equation $\ddot{x}+x=f(x, t^{-1/2})$.

W. R. Wasow (Los Angeles, Calif.).

Taam, Choy-Tak. The boundedness of the solutions of a differential equation in the complex domain. Pacific J. Math. **2**, 643–654 (1952).

Estimates are given for the rate of growth of solutions of $d^2W/dx^2+Q(x)W=0$ as $x\to\infty$, in terms of

$$\phi(x)=\int_0^x(|q_1(t)-a|+|q_2(t)|)dt\quad(Q=q_1-iq_2, a>0);$$

the results generalise those of Levinson [Duke Math. J. **8**, 1–10 (1941); these Rev. **2**, 287] for Q real. If $\phi(x)=O(1)$, it is shown that $W(x)=A\sin a^{1/2}x+B\cos a^{1/2}x+o(1)$. This can be extended, by Phragmén-Lindelöf arguments, to solutions of $d^2W/ds^2+Q(z)W=0$ in a half-strip $x\geqq 0$, $\alpha\leqq y\leqq\beta$ $(s=x+iy)$. *G. E. H. Reuter* (Manchester).

Yakubovič, V. A. An estimate of the characteristic exponents and criteria of stability for a linear differential equation of the second order with periodic coefficients. Doklady Akad. Nauk SSSR (N.S.) **87**, 345–348 (1952). (Russian)

Given the equation $u''+p(t)u=0$, where $p(t)$ is periodic, the author obtains various conditions which ensure $u=O(e^{\epsilon t})$ as $t\to\infty$. *R. Bellman* (Santa Monica, Calif.).

Haacke, Wolfhart. Über die Stabilität eines Systems von gewöhnlichen linearen Differentialgleichungen zweiter Ordnung mit periodischen Koeffizienten, die von Parametern abhängen. II. Math. Z. **57**, 34–45 (1952).

In the system of differential equations

$$r^2y''+Qy+\epsilon A(x, \epsilon)y=0 \tag{1}$$

y denotes an n-dimensional column vector, r and ϵ are positive parameters, and Q is a constant, real diagonal matrix. The real matrix $A(x, \epsilon)$ is an even, continuous, periodic function of x with period 2π. Also, $A(x, \epsilon)$ is analytic in ϵ for $|\epsilon|\leqq E_0$ and all real x. The problem under consideration is to study the regions of the (r, ϵ)-plane in which (1) is stable in the sense that all its solutions are uniformly bounded for all real x. As in the first part of this paper [Math. Z. **56**, 65–79 (1952); these Rev. **14**, 276] a fundamental system of (1) is found in the form of series in powers of a parameter, here ϵ. But these series are here convergent rather than asymptotic, with a radius of convergence for which lower bounds can be derived from a result of Perron [Math. Ann. **113**, 292–303 (1936)]. These series permit the study of the characteristic roots of (1) as in part I. If Q has some negative elements, the system turns out to be unstable for small ϵ. If Q is positive definite, the curves separating the stable from the unstable regions of the (r, ϵ)-plane intersect the r-axis in an infinite set of points that are calculated explicitly. For $n=2$ a more detailed analysis is given. *W. R. Wasow* (Los Angeles, Calif.).

Barbašin, E. A., and Krasovskiĭ, N. N. On stability of motion in the large. Doklady Akad. Nauk SSSR (N.S.) **86**, 453–456 (1952). (Russian)

The authors consider differential equations (1) $\dot{x}=X(x)$, where x is a vector in n-dimensional space and the components X_i of X are of class C' everywhere, with $X(0)=0$. The solution $x\equiv 0$ is said to be asymptotically stable for arbitrary initial conditions if (*) every solution $x(t)$ of (1) tends to 0 as $t\to+\infty$. Existence of a differentiable function $v(x)$, everywhere positive, such that $\dot{v}<0$ on each solution would, according to Liapounoff's theory, guarantee a local stability of the solution $x\equiv 0$. It is shown by an example that this need not imply (*). If, in addition, $v(x)\to\infty$ as $\|x\|\to\infty$, then it is shown that (*) does follow. Conversely, if (*) holds, then such a v does exist. If the condition $\dot{v}<0$ is replaced by the conditions: $\dot{v}<0$ outside of M, $\dot{v}\leqq 0$ in M, and M is a set such that the intersection of each level surface $v=c$ and M contains no positive semi-trajectory of (1), then (*) again follows. It is shown that these hypotheses are satisfied by the system of second order: $\dot{x}=y$, $\dot{y}=-\phi(y)-g(y)f(x)$, provided $f(0)=\phi(0)=0$, $xf(x)>0$ for $x\neq 0$, $y\phi(y)>0$ for $y\neq 0$, $g(y)>0$, $\int_0^x f(x)dx\to\infty$ as $|x|\to\infty$ and $\int_0^y yg(y)^{-1}dy\to\infty$ as $y\to\infty$. In the proofs use is made of results of a previous paper by Barbašin [Mat. Sbornik N.S. **29(71)**, 233–280 (1951); these Rev. **13**, 756].

W. Kaplan (Ann Arbor, Mich.).

Cartwright, M. L. Van der Pol's equation for relaxation oscillations. Contributions to the Theory of Nonlinear Oscillations, vol. II, pp. 3–18. Princeton University Press, Princeton, 1952.

The author treats the differential equation

$$\ddot{x}-k(1-x^2)\dot{x}+x=0$$

of van der Pol and obtains an asymptotic formula for the period T of the periodic solution of the equation for large k, and also a formula for the maximum velocity $V_{\max}$ in such a periodic motion. The formulas are:

$$T=2k(\tfrac{3}{2}-\log_e 2)+\frac{3(\alpha+\beta)}{k^{1/3}}+O\left(\frac{1}{k^{1/3}}\right),$$

$$V_{\max}=2+\frac{\alpha+\beta}{3k^{4/3}}+O\left(\frac{1}{k^{4/3}}\right).$$

The constants α and β are determined as follows: The differential equation $\eta_0 d\eta_0/d\xi=2\xi\eta_0+1$ has exactly one solution $\eta_0^*(\xi)$ which tends to zero as $\xi\to-\infty$, and the constants α and β are given in terms of this solution: $\alpha=\eta_0^*(0)$, $\beta=\int_0^\infty d\xi/\eta_0^*(\xi)$. Complete asymptotic developments valid for k large for the same differential equations have been given by Dorodnitsin [Akad. Nauk SSSR. Prikl. Mat. Meh. 11, 313–328 (1947); these Rev. 9, 144], and for much more general differential equations by J. Haag [Ann. Sci. Ecole Norm. Sup. (3) 60, 35–64, 65–111 (1943); 61, 73–117 (1944); these Rev. 7, 299]. The methods used by the author are quite different from those used by Dorodnitsin and Haag; they are too complicated to describe in a brief review.

J. J. Stoker (New York, N. Y.).

Klotter, K., and Pinney, E. A comprehensive stability criterion for forced vibrations in nonlinear systems. J. Appl. Mech. 20, 9–12 (1953).

The authors examine the stability of the forced periodic solutions of the differential equation

$$\ddot{q}+k^2q+k^2\Phi(q,\dot{q})=p\cos\Omega t \tag{1}$$

by the averaging method; i.e., they introduce new variables Q and θ by means of the transformation $q=Q\cos(\Omega t+\theta)$, $\dot{q}=-kQ\sin(\Omega t+\theta)$, derive the system of differential equations for Q and θ, and consider the "averaged" system

$$\begin{aligned}\dot{Q}/k&=S(Q)-(p/2k^2)\sin\theta,\\ Q\dot{\theta}/k&=C(Q)-(p/2k^2)\cos\theta+Q\Delta,\end{aligned} \tag{2}$$

where $\Delta=1-\Omega/k$,

$$S(Q)=\frac{1}{2\pi}\int_0^{2\pi}\Phi(Q\cos x,\ -kQ\sin x)\sin x\,dx$$

and

$$C(Q)=\frac{1}{2\pi}\int_0^{2\pi}\Phi(Q\cos x,\ -kQ\sin x)\cos x\,dx.$$

A singular point, (Q_0,θ_0), of the system (2) leads to an approximation to a periodic solution of (1), which is considered stable or unstable according as the corresponding singular point of (2) is stable or unstable. Thus the authors conclude that $B(Q_0)=S(Q_0)+Q_0S'(Q_0)>0$ implies instability of the periodic solution while if $B(Q_0)\leqq 0$, then $dp/dQ_0>0$, $dp/dQ_0<0$ imply stability and instability respectively. If $q^*=\max q(t)$ for the periodic solution these last conditions become $\partial q^*/\partial p>0$ for stability and $\partial q^*/\partial p>0$ for instability.

The authors' method of attributing to a periodic solution of (1) the same stability characteristics as possessed by the corresponding singular point of (2) can be justified (except perhaps when $B(Q_0)=0$) by the perturbation method, at least for sufficiently small Φ and p.

C. E. Langenhop.

Zlámal, Miloš. Nonlinear forced oscillations. Časopis Pěst. Mat. 77, 53–64 (1952). (Czech)

This is an expository paper on the differential equation $\ddot{x}+f(x,\dot{x})\dot{x}+g(x)=p(t)$, $p(t)$ periodic. The author describes some results achieved between 1940 and 1950.

A. Erdélyi (Pasadena, Calif.).

Seifert, George. On the existence of certain solutions of a nonlinear differential equation. Z. Angew. Math. Physik 3, 468–471 (1952).

L'autore studia l'equazione

$$\frac{d^2\theta}{dt^2}+\alpha\frac{d\theta}{dt}+\sin\theta=\beta \tag{1}$$

($\alpha>0$, $0<\beta<1$), migliorando precedenti risultati di Tricomi [Ann. Scuola Norm. Super. Pisa (2) 2, 1–20 (1933)] e di Amerio [Ann. Mat. Pura Appl. (4) 30, 75–90 (1949); questi Rev. 11, 723] circa la determinazione del valore critico α_0 tale che per $\alpha\leqq\alpha_0$ la (1) ammetta una soluzione $\theta(t)$ con $\theta'(t)$ periodica, mentre per $\alpha>\alpha_0$ tale soluzione non esista.

L. Amerio (Milano).

Cohn, George I., and Saltzberg, Bernard. Solution of nonlinear differential equations by the reversion method. J. Appl. Phys. 24, 180–186 (1953).

Equations of the type $Z_1I+Z_2I^2+\cdots+Z_nI^n=\lambda V$, where $I(t)$ is the unknown function, $V(t)$ is known, Z_i represent differential and integral operators, and λ is a parameter, are considered. The Van Orstrand procedure [cf. L. A. Pipes, J. Appl. Phys. 23, 202–207 (1952); these Rev. 13, 783] of obtaining approximate solutions by substituting $\sum_{m=1}^{n}\lambda^m I_m(t)$ for $I(t)$ and identifying coefficients of like powers of λ to get recurrence equations in the functions $I_m(t)$ is systematized so as to extend the tabulation of these recurrence equations. Two examples are carried out as a test of the approximation.

R. V. Churchill (Ann Arbor, Mich.).

Combes, Jean. Sur une classe d'équations différentielles d'ordre infini. Ann. Fac. Sci. Univ. Toulouse (4) 15, 195–205 (1951).

The equations of the title are of type

$$\sum_{n=0}^{\infty}c_n y^{(n)}(x)=z(x), \tag{1}$$

with constant coefficients $\{c_n\}$ that satisfy the condition (2) $c_n>0$, c_{n+1}/c_n non-decreasing; and with given analytic function $z(x)$. It is shown that $\sum_0^\infty c_n y^{(n)}(x)$ converges at one point (which can be taken as $x=0$) if and only if $\sum_0^\infty c_nA_n$ converges, where $y(x)=\sum_0^\infty a_nx^n$ and $A_n=n!a_n$; and in this case the series in (1) converges for all x (uniformly in every bounded region), and $y(x)$ (and $z(x)$ also if (1) holds) is an entire function of order $\rho\leqq 1$. Let $z(x)=\sum_0^\infty b_nx^n$ be entire, and set $B_n=n!b_n$. Equation (1) is equivalent to the infinite system of linear equations (3) $\sum_{k=0}^{\infty}c_kA_{n+k}=B_n$ ($n=0,1,\cdots$). If $z(x)\equiv 0$, i.e., $B_n=0$, then (3) has only the trivial solution $A_n=0$; so (3) cannot possess more than one solution. For the case $B_n\neq 0$, necessary conditions for a solution of (3) are $c_nB_n\to 0$ and $|\sum_{k=0}^{n}\gamma_kB_{n+k}|<\infty$ ($n=0,1,\cdots$). Here $\{\gamma_n\}$ are the constants defined by $A_0=\sum_{p=0}^{n}\gamma_pB_p$ when system (3) is replaced by the reduced system of order $n+1$: $\sum_{k=0}^{n}c_kA_{s+k}=B_s$ ($s=0,1,\cdots,n$; $n=0,1,\cdots$), and the resulting system is solved for $A_0,\cdots,A_n$. A sufficient condition for a solution is $\sum_0^\infty|c_nB_n|<\infty$. If there is a p for

which $\sum_{n=0}^{\infty} c_n/c_{n+p} < \infty$, then $|\sum_0^\infty c_n B_n| < \infty$ is necessary and sufficient for (3) to have a solution.

Finally, let $y(x)$ be an entire function of order not exceeding one, and consider sequences $\{c_n\}$ of real or complex numbers for which $|c_{n+1}/c_n|$ is non-decreasing and such that $\lim_{n\to\infty} \log|c_n|/n \log n$ exists, $=1/\rho-1$. The number ρ is called the index of the sequence $\{c_n\}$. The order of y is then equal to lim sup $\{\rho\}$ where ρ ranges over the set of indices of all sequences $\{c_n\}$ for which $\sum_0^\infty c_n y^{(n)}(x)$ converges.

I. M. Sheffer (State College, Pa.).

Dolgolenko, Yu. V. Stability and auto-oscillations of a class of relay systems of automatic discontinuous regulation. Akad. Nauk SSSR. Inženernyĭ Sbornik **13**, 161–176 (1952). (Russian)

The systems considered are closed relay circuits whose only nonlinear element is a pulsed limiter whose output $y(t)$ for input $x(t)$ is given by $y(t)=k$ sign $x(t)$ for $nT<t<(n+\gamma)T$ and $y(t)=0$ for $(n+\gamma)T<t<(n+1)T$ $(n=0, 1, \cdots)$. Let $u(t/T)$ be the steady-state output of the circuit opened at the limiter for the input $x(t)=1$ for $2NT<t<(2N+1)T$, $x(t)=-1$ for $(2N+1)T<t<(2N+2)T$ $(N=0, 1, \cdots)$. The author calls $u(N)$ the characteristic of the open system and shows how it can be used to decide whether the regulator is unstable or generates stable auto-oscillations. *M. Golomb* (Lafayette, Ind.).

Bückner, Hans. Inequalities for solutions of linear differential equations. A contribution to the theory of servomechanisms. Edinburgh Math. Notes **38**, 13–16 (1952).

Let $\sigma_k \pm i\tau_k$ $(k=1, \cdots, m)$ be the nonreal characteristic exponents for the differential equation

$$y+c_1y'+\cdots+c_ny^{(n)}=f(x),$$

and assume no σ_k is 0, $f(x)$ is continuous and $|f(x)| \leqq M$ for $a \leqq x \leqq b$. It is proved by the author that under these assumptions the equation has a particular solution $\eta(x)$ for which $|\eta(x)| \leqq CM$ for $a \leqq x \leqq b$, where $C=\prod \coth \frac{1}{2}\pi|\sigma_k/\tau_k|$ $(C=1$ when $m=0)$. Some applications are pointed out.

M. Golomb (Lafayette, Ind.).

Alzerman, M. A. On the construction of resonance graphs for systems with nonlinear feedback. Akad. Nauk SSSR. Inženernyĭ Sbornik **13**, 151–160 (1952). (Russian)

Consider the equation $D(p)x=K(p)[f(x)+A \sin \omega t]$, where $p=d/dt$, $D(p)$ and $K(p)$ are polynomials with constant coefficients, $f(x)$ a given function. The values of A, ω are assumed to be such that this equation has a periodic solution close to the harmonic $x=d+c \sin(\omega t-\gamma)$. If $f(d+c \sin u)$ is approximated by

$$f_0(c, d)+a(c, d) \sin u+b(c, d) \cos u,$$

equations for c, d, γ result which the author solves by a construction based on the graphs of $D(i\omega)/K(i\omega)$ $(0<\omega<\infty)$ and of $[a(c, d)+ib(c, d)]/c$ $(0<c<\infty)$. Thus c, d, γ are obtained as functions of A, ω. A similar construction can be used to find c, γ, C, Ω for an approximate solution of the form $x=c \sin(\omega t-\gamma)+C \sin \Omega t$, where Ω is the frequency of the auto-oscillations. *M. Golomb* (Lafayette, Ind.).

Bautin, N. N. A dynamical model of a chronometric movement. Akad. Nauk SSSR. Inženernyĭ Sbornik **12**, 3–22 (1952). (Russian)

A simplified model of a complicated clock mechanism is given consisting of a mass attached to a spring in rectilinear motion and undergoing impulses from a moving infinite band from which protrude teeth at equally spaced intervals. The motion is reduced to the solution of four linear differential equations with a discontinuity. The solutions are in terms of two parameters ξ, η which undergo a linear transformation S. The fixed points of S correspond to the periodic motions of the initial mechanism and their stability is discussed. [References: Andronov and Naĭmark, C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 17–20 (1946); these Rev. **8**, 101; Bautin, ibid. **61**, 17–20 (1948); **65**, 279–282 (1949); **72**, 19–22 (1950); these Rev. **10**, 409, 748; **11**, 597.]

S. Lefschetz (Princeton, N. J.).

Popov, B. S. Factorization of an operator. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **4**, no. 7, 20 pp. (1951). (Macedonian. English summary)

It is known that the hypergeometric equation

$$x(x-1)y''+[(\alpha+\beta+1)x-\gamma]y'+\alpha\beta y=0$$

is reducible if and only if one of the numbers α, β, $\gamma-\alpha$, $\gamma-\beta$ is a positive or negative integer. The author finds an explicit reduction by determining the polynomials $P_i(x)$ for the equivalent equation

$$[P_1(x)d/dx+1][P_2(x)d/dx+P_3(x)]y=0.$$

M. Golomb (Lafayette, Ind.).

Popov, B. S. Remarque sur l'équation de Riccati. Bull. Soc. Math. Phys. Macédoine **2**, 113–115 (1951). (Macedonian. French summary)

It is shown that if the coefficients of the Riccati equation satisfy a certain condition, then the equation can be integrated without any quadratures. *M. Golomb.*

Kito, Fumiki. On a property of linear ordinary differential equation which relates to "end effect" in theory of curved shells. Proc. Fac. Engrg. Keiogijuku Univ. **3**, no. 8, 16–21 (1950).

In the differential equation

$$\mu\left[\frac{d^nu}{dt^n}+p_1(t)\frac{d^{n-1}u}{dt^{n-1}}+\cdots+p_s(t)\frac{d^{n-s}u}{dt^{n-s}}\right]+p_{s+1}(t)\frac{d^{n-s-1}u}{dt^{n-s-1}}+\cdots+p_n(t)u=q(t)$$

the coefficients are continuous real functions, and μ is a small parameter. By means of an operational technique [cf. V. Volterra and J. Pérès, Leçons sur la composition et les fonctions permutables, Gauthier-Villars, Paris, 1924] the solution assuming fixed initial values for $t=\tau$ is represented in a form which sets in evidence the rapidly changing exponential terms that are present for small μ.

W. R. Wasow (Los Angeles, Calif.).

Bakaev, Yu. N. Approximate integration of the differential equation of a pendulum. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 723–728 (1952). (Russian)

The equation $\ddot{v}+\alpha\dot{v}+\sin v=\beta$ is proposed for study. For $\alpha^2 \gg 1$, the author gives a solution by the method of small parameters, and he approximates the solution for small damping (and with $\beta=0$) by assuming small change in amplitude over a given period. The intermediate case is discussed only qualitatively. It is shown that there exists a value $\alpha_*=f(\beta)$ such that for $\alpha>\alpha_*$, the motion approaches an equilibrium value, while if $\alpha<\alpha_*$, the motion may be periodic or not, depending on initial conditions. [See

Andronow and Chaikin, Theory of oscillations, Princeton, 1949, pp. 287–300; these Rev. 10, 535.] *R. E. Gaskell.*

Huber, A. Das Verhalten der Integrale der Gibbs-Duhem-Marguels'schen Gleichung für binäre Gemische in der Umgebung ihrer festen singulären Stellen. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **160**, 181–197 (1951).

The equation in question is

$$\frac{dp(x)}{dx}=\frac{p(x)(1-x)P'(x)}{p(x)-xP(x)},$$

where x is the mole fraction of one component, $p(x)$ is the partial pressure of that component, and $P(x)$ is the total pressure in the binary mixture. The author recognizes three singular points of this equation and obtains their Poincaré classifications. This provides qualitative information as to the behavior of $p(x)$, which is illustrated by several graphs. The special cases considered are too numerous to list here.

E. Pinney (Berkeley, Calif.).

Mikeladze, Š. E. A new integral method of solution of boundary problems. Soobščeniya Akad. Nauk Gruzin. SSR **12**, 393–396 (1951). (Russian)

In order to obtain the piecewise continuous solution $y(x)$, with piecewise continuous derivatives, of a linear differential equation the author uses a development

$$y(x)=\sum_{\nu=0}^{n-1}\frac{y^{(\nu)}(0)}{\nu!}x^\nu+\sum_m A_m+\sum_r B_r(x-a_r)+\cdots$$
$$+\sum_\mu Q_\mu(x-a_\mu)x^{n-1}+\frac{1}{(n-1)!}\int_0^x(x-t)^{n-1}y^{(n)}(t)dt.$$

He substitutes in the last integral the nth derivative as given by the differential equation, eliminates, by partial integration, all derivatives of $y(x)$ and in this way transforms boundary problems to integral equations analogous to the equation of Volterra in the case of continuous functions.

E. M. Bruins (Baghdad).

Raševskiĭ, P. K. On the solution of boundary problems by methods of noncommutative algebra. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk **1950**, no. 9, 3–12 (1950). (Russian)

An algebraic formalism is employed to obtain conditions on a, b, c which are necessary and sufficient for the existence of a solution of the boundary-value problem

$$(1-y^2)[(1-y^2)f'(y)]'+(ay^2+by+c)f(y)=0,$$
$$f(-1)=f(1)=0.$$

The operators y: $f(y)\to yf(y)$, and p: $f(y)\to i(1-y^2)f'(y)$, generate a noncommutative associative algebra of operators over the complex field, and the differential equation takes the form $(p^2-ay^2-by-c)f(y)=0$. The discussion is carried through by manipulation of theo perator in the parentheses.

J. G. Wendel (Baton Rouge, La.).

Yoshizawa, Taro. Note on the non-increasing solutions of $y''=f(x,y,y')$. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **27**, 153–162 (1952).

L'A. studia problemi al contorno per l'equazione

$$y''=f(x,y,y').$$

Suppone che $f(x,y,z)$ sia continua nell'insieme R: $x\geqq 0$, $y\geqq 0$, $z\leqq 0$; che $f(x,0,0)$ sia identicamente nulla e che $f(x,y,0)$ non sia mai positiva. Suppone che dato ad arbitrio il numero positivo C si possano determinare la costante negativa $-K$ e le funzioni $\Phi(x,y,z)$ e $\Psi(x,y,z)$ in guisa che $\Phi(x,y,z)$ e $\Psi(x,y,z)$ siano continue positive e uniformemente infinitesime all'infinito nell'insieme R_C: $0\leqq x\leqq C$, $0\leqq y\leqq C$, $z\leqq -K$, siano dotate dei derivate prime continue nell'interno di R_C e soddisfacciano alle $\Phi_x'+\Phi_y'z+\Phi_z'f\geqq 0$ e $\Psi_x'+\Psi_y'z+\Psi_z'f\leqq 0$. E dimostra che allora l'equazione $y''=f(x,y,y')$ ammette almeno una soluzione che assuma un valore positivo, prefissato a piacere, nel punto $x=0$ e sia nulla per $x=x_0$, con x_0 positivo e prefissato a piacere; che l'equazione stessa ammette almeno una soluzione che assuma un valore positivo prefissato a piacere nell'origine, sia definita per ogni valore non negativo della x ed abbia una derivata giammai positiva. L'A. indica altresì condizioni sufficienti perchè quest'ultima soluzione sia infinitesima all'infinito. I risultati dell'A. estendono alcuni teoremi di Hartman e Wintner [Amer. J. Math. **73**, 390–404 (1951); questi Rev. **13**, 37]. *G. Scorza Dragoni.*

Avakumović, Vojislav G. Sur le problème aux limites des équations différentielles du second ordre non linéaires. Bull. Acad. Serbe Sci. (N.S.) **5**, Cl. Sci. Math. Nat. Sci. Math. **1**, 183–187 (1952).

It is proved that the usual successive approximations converge to the unique solution of the problem $y''=f(x,y,y')$, $y(a)=A$, $y(b)=B$, if $f(x,y,y')$ satisfies a certain generalized Lipschitz condition together with a growth condition.

M. Golomb (Lafayette, Ind.).

Kreĭn, M. G. On inverse problems for a nonhomogeneous cord. Doklady Akad. Nauk SSSR (N.S.) **82**, 669–672 (1952). (Russian)

The author presents a rather concentrated summary of his further progress in the theory of a cord stretched to unit tension between $x=0$ and $x=1$, particularly the determination of the mass-distribution $d\sigma(x)$ from the characteristic frequencies and other dynamical properties [see M. G. Kreĭn, same Doklady (N.S.) **76**, 345–348 (1951); these Rev. **13**, 43]. Let $\lambda_0<\lambda_1<\cdots$ be the characteristic numbers when the cord (then referred to as S_2) is fixed at both ends, $\mu_0<\mu_1<\cdots$ the same for the cord (S_1) fixed at the right end only, the left end being free to move transversely. The first problem is the determination of $\sigma(x)$ from the λ_r. It is necessary and sufficient that $\lim n/\lambda_n^{1/2}$ exist and be finite, and that $\sum_0^\infty \lambda_n^{-2}|D'(\lambda_n)|^{-1}<\infty$, where $D(\lambda)=\prod_0^\infty(1-\lambda/\lambda_n)$; then among all such S_2 with the given spectrum there is a unique symmetrical one, which is also that of least total mass. This confirms a previous conjecture [loc. cit.]; theorem 2 of the previous paper gave slightly different conditions. Next studied is the effect of a pulsating force $A\sin\lambda^{1/2}t$ applied to the free end of S_1, leading to the "coefficient of dynamical pliability" $R(\lambda)=D(\lambda)/D_1(\lambda)$, where $D_1(\lambda)=\prod_0^\infty(1-\lambda/\mu_n)$ (cf. $E(\lambda)/D(\lambda)$ of §4 of the cited paper); $\sigma(x)$ is determined by $R(\lambda)$. Regarding the determination of $\sigma(x)$ from the μ_r, it is necessary and sufficient that the limit $T=\pi\lim(n/\mu_n^{1/2})$ exist and be finite, and that $\sum_0^\infty \mu_n^{-3/2}|D_1'(\mu_n)|^{-1}<\infty$. There is then a unique S_1 with minimum mass. The remainder of the paper is concerned with the properties of a "transition function" $\Phi(t)$, i.e., the displacement of the free end of S_1 as a result of unit force applied at the free end for time t. The theorems give conditions under which $\sigma(x)$ is determined by $\Phi(t)$, and conditions for a given $\Phi(t)$ to be a "transition function" for certain types of cord. *F. V. Atkinson* (Ibadan).

Krein, M. G. On some new problems of the theory of oscillations of Sturmian systems. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 555–568 (1952). (Russian)

The author now [cf. preceding review] solves in detail the inverse problem for the light string bearing particles of masses $m_1, \cdots, m_n$, the solubility conditions being automatically satisfied in the case of a finite number of degrees of freedom. For the string with one end fixed and given spectrum $\lambda_1<\lambda_2<\cdots<\lambda_n$, the minimum total mass is $M_m=\{\sum_1^n\lambda_j^{-3/2}|P'(\lambda_j)|^{-1}\}^2$ where $P(\lambda)=\prod_1^n(1-\lambda/\lambda_j)$, which is uniquely attained by a system with a certain specified "coefficient of dynamical pliability". Analogous results are given for the cases of both ends fixed, both ends free to move transversely. The solution makes use of continued fractions. The author concludes with a rather general discussion of the case of an infinite number of particles.

F. V. Atkinson (Ibadan).

Lukomskaya, M. A. On cycles of systems of linear homogeneous differential equations. Mat. Sbornik N.S. **29(71)**, 551–558 (1951). (Russian)

A. Markuševič observed that every system of linear partial differential equations

$$(T_0) \qquad c_iu_x+d_iv_x=a_iu_y+b_iv_y, \quad i=1, 2,$$

with sufficiently smooth coefficients $a_1(x, y), \cdots, d_2(x, y)$ can be written in a form such that $\partial c_i/\partial x=\partial a_i/\partial y$, $\partial d_i/\partial x=\partial b_i/\partial y$, $i=1, 2$. In this case the integrals

$$U=\int(a_2u+b_2v)dx+(c_2u+d_2v)dy,$$

$$V=\int(a_1u+b_1v)dx+(c_1u+d_1v)dy,$$

are path-independent and (u, v) satisfy a system (T_1) which is of the same form as (T_0). [Cf. I. Petrovskiĭ, Uspehi Matem. Nauk (N.S.) **1**, no. 3–4 (13–14), 44–70 (1946); these Rev. **10**, 301.] System (T_0) is said to be embedded into a cycle if there exists a sequence of systems (T_0), (T_1), (T_2), $\cdots$ such that (T_r) is related to (T_{r+1}) as (T_0) was related to (T_1). The cycle is called of finite order n if (T_n) is equivalent to (T_0), of infinite order if there is no such n. The author (a) proves that every (T_0) can be embedded into a cycle of infinite order, and (b) gives necessary and sufficient conditions in order that the minimum order $n_{\min}$ of a cycle beginning with (T_0) be 1. She states as an open problem the question on the existence of systems with $n_{\min}>2$.

The reviewer remarks that for elliptic systems the natural setting for this problem is the theory of pseudo-analytic functions [L. Bers, Proc. Nat. Acad. Sci. U. S. A. **37**, 42–47 (1951); these Rev. **13**, 352]. Using the formalism of pseudo-analytic functions one obtains the result (a), (b) almost at once and the 10 conditions in (b) stated by the author can be replaced by one [cf. L. Bers, Theory of pseudo-analytic functions, mimeographed lecture notes, New York University, 1953]. Recently M. H. Protter proved the possibility of $n_{\min}>2$ (oral communication). *L. Bers.*

Volpato, Mario. Un criterio di confronto per le soluzioni di una equazione alle derivate parziali del primo ordine. Ann. Univ. Ferrara. Sez. VII. (N.S.) **1**, 127–133 (1952).

Let two solutions of $z_x=f(x, y, z, z_y)$ have a common domain D of definition. Conditions are given sufficient to insure that if $z_1(x_0, y)\neq z_2(x_0, y)$ for some fixed x_0 and all (x_0, y) in D, then $z_1(x, y)\neq z_2(x, y)$ for all (x, y) in D.

F. A. Ficken (Knoxville, Tenn.).

Saltykow, N. Ordre d'un système d'équations différentielles ordinaires de la forme générale. Bull. Acad. Serbe Sci. (N.S.) **5**, Cl. Sci. Math. Nat. Sci. Math. **1**, 141–148 (1952).

This paper is identical in content with one reviewed earlier [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **37**, 213–226 (1951); these Rev. **13**, 131]. *M. Golomb.*

Saltykow, N. Généralisation par S. Lie de la théorie du dernier multiplicateur. Bull. Acad. Serbe Sci. (N.S.) **5**, Cl. Sci. Math. Nat. Sci. Math. **1**, 1–9 (1952).

This paper is identical in content with one reviewed earlier [Glas Srpske Akad. Nauka. Od. Prirod.-Mat. Nauka **198**, 1–16 (1950); these Rev. **12**, 829]. *M. Golomb.*

Saltykow, N. Problèmes d'intégration des équations aux différentielles totales. Bull. Acad. Serbe Sci. (N.S.) **5**, Cl. Sci. Math. Nat. Sci. Math. **1**, 89–100 (1952).

This paper is identical in content with one reviewed earlier [Glas Srpske Akad. Nauka. Od. Prirod.-Mat. Nauka **198**, 87–104 (1950); these Rev. **12**, 829]. *M. Golomb.*

Saltykow, N. Note sur l'intégration des équations linéaires aux dérivées partielles du premier ordre admettant les transformations infinitésimales. Bull. Acad. Serbe Sci. (N.S.) **5**, Cl. Sci. Math. Nat. Sci. Math. **1**, 29–35 (1952).

This paper is identical in content with one reviewed earlier [Glas Srpske Akad. Nauka. Od. Prirod.-Mat. Nauka **198**, 53–62 (1950); these Rev. **12**, 829]. *M. Golomb.*

Saltykow, N. Domaine d'existence des intégrales d'un système d'équations aux dérivées partielles d'ordres supérieurs au premier. Bull. Acad. Serbe Sci. (N.S.) **5**, Cl. Sci. Math. Nat. Sci. Math. **1**, 149–156 (1952).

The author applies his modification of the majorant method [cf. Publ. Math. Univ. Belgrade **6–7**, 189–196 (1938)] to the existence proof of analytic solutions for systems of partial differential equations. *M. Golomb.*

Saltykow, N. Invariants différentiels des groupes fonctionnels d'intégrales. Bull. Acad. Serbe Sci. (N.S.) **5**, Cl. Sci. Math. Nat. Sci. Math. **1**, 11–28 (1952).

This paper is identical in content with one reviewed earlier [Glas Srpske Akad. Nauka. Od. Prirod.-Mat. Nauka **198**, 17–35 (1950); these Rev. **12**, 829]. *M. Golomb.*

Saltykow, N. Théories analytiques et géométriques des équations aux dérivées partielles du premier ordre. Bull. Acad. Serbe Sci. (N.S.) **5**, Cl. Sci. Math. Nat. Sci. Math. **1**, 127–128 (1952).

Verfasser gibt einen historischen Bericht über die Entwicklung der Integrationstheorie der partiellen Differentialgleichungen erster Ordnung. Es handelt sich um Probleme, die bereits von G. Monge gestellt worden sind. Verfasser erwähnt aus der neuren Zeit insbesondere die Untersuchungen von H. Léauté [Dissertation, Paris, 1876], Hadamard [Cours d'analyse, t. II, Hermann, Paris, 1930] und Rachajsky [siehe nachstehendes Referat], mit welchen ein gewisser Abschluss der Untersuchungen erreicht wird.

M. Pinl (Dacca).

Rachajsky, B. Les fonctions caractéristiques dans la théorie géométrique des caractéristiques pour les équations aux dérivées partielles du premier ordre. Bull. Acad. Serbe Sci. (N.S.) **5**, Cl. Sci. Math. Nat. Sci. Math. **1**, 129–139 (1952).

Die partielle Differentialgleichung erster Ordnung

$$F(x, y, z, p, q) = 0$$

integrieren heisst in allgemeinster Weise fünf Funktionen der unabhängigen Parameter α und β bestimmen, welche $F=0$ in eine Identität in α und β verwandeln. Cauchy gelangte zu der nach ihm benannten Charakteristikentheorie zunächst durch eine spezielle bereits von Ampère verwendete Transformation, welche x als Parameter beibehält, y jedoch als Funktion von x und u einführt, derart dass ein System von vier Gleichungen entsteht, in welchem nur partielle Ableitungen nach x auftreten:

$$P\frac{\partial y}{\partial x} = Q, \quad P\frac{\partial z}{\partial x} = Py + Qq, \quad P\frac{\partial q}{\partial x} = -Y - qZ, \quad P\frac{\partial p}{\partial x} = -X - pZ.$$

Bedeutet dann U den Ausdruck $\partial z/\partial u - q\partial y/\partial u$, so besteht die Beziehung

$$(*) \qquad U = U_0 \exp\left(-\int \frac{Z}{P}dx\right).$$

Eine geometrische Deutung der Beziehung (*) hat bereits H. Léauté [Dissertation, Paris, 1876] gegeben. Verf. stellt sich die Aufgabe, die Voraussetzungen zu untersuchen, unter welchen diese geometrische Deutung gilt. Es ergeben sich Bedingungen für die Charakteristiken, die sich durch charakteristische Funktionen ausdrücken lassen. Die Untersuchung wird für vollständige Integrale erster und zweiter Klasse sowie auch in Fällen mit beliebigen willkürlichen Konstanten durchgeführt. *M. Pinl* (Dacca).

Saltykow, N. Méthodes d'intégration des équations aux dérivées partielles du second ordre à une fonction inconnue. Bull. Acad. Serbe Sci. (N.S.) **5**, Cl. Sci. Math. Nat. Sci. Math. **1**, 101–122 (1952).

This paper is identical in content with one reviewed earlier [Glas Srpske Akad. Nauka. Od. Prirod.-Mat. Nauka **198**, 37–52 (1950); these Rev. **12**, 829]. *M. Golomb.*

Ciliberto, Carlo. Su di un problema al contorno per l'equazione $u_{xx} - u_y = f(x, y, u, u_x)$. Ricerche Mat. **1**, 295–316 (1952).

Let the continuous functions $f_i(y)$ $(i=1, 2)$ satisfy for y in $0 \leqq y \leqq b$ the conditions

$$f_i(0) = 0, \quad |f_i(y_2) - f_i(y_1)| \leqq |y_2 - y_1|^{(1+\gamma)/2}, \quad \gamma < 1.$$

Further assume $l(y) \geqq 0$ and $m(y) \geqq 0$ are bounded non-decreasing functions of y; and that $f(x, y, z, w)$ satisfies a Hölder condition of exponent λ with respect to the variables x, y, z, w as well as the condition

$$|f(x, y, z, w)| \leqq l(y)[|z| + |w|] + m(y).$$

Under these restrictions, together with a certain bound on $l(b)$, there is shown to exist at least one continuous function $v(x, y)$ satisfying the conditions:

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial y} = f(x, y, v, v_x) \quad \text{for } 0 < x < a,\ 0 < y < b,$$

$$v(0, y) = f_1(y), \quad v(a, y) = f_2(y), \quad v(x, 0) = 0.$$

As in an earlier paper in which $f(x, y, v, v_x)$ was independent of v_x the author makes use of a Brouwer fixed point theorem in his existence proof [Ricerche Mat. **1**, 55–77 (1952); these Rev. **14**, 51]. *F. G. Dressel* (Durham, N. C.).

Gandin, L. S., and Soloveĭčik, R. È. On the theory of evaporation for a bounded water-basin. Doklady Akad. Nauk SSSR (N.S.) **86**, 55–57 (1952). (Russian)

Dans l'étude de ce problème les prédecesseurs de Gandin and Soloveĭčik n'ont pas tenu suffisamment compte de plusieurs facteurs importants: on a supposé sans raison le coefficient d'échange indépendent d'altitude, on a donné une forme insuffisamment générale aux conditions à la limite et on n'a pas tenu compte de brassage turbulent horizontal. Les auteurs donnent à l'équation fondamental la forme

$$(1) \qquad u\frac{\partial q}{\partial x} = K_x\frac{\partial^2 q}{\partial x^2} + K_y\frac{\partial^2 q}{\partial y^2} + \frac{\partial}{\partial z}\left(K_z\frac{\partial q}{\partial z}\right)$$

avec q = humidité spécifique, $u = u_1 z^n$ vitesse de vent (dirigé suivant l'axe X), $K_x = K_{x1}z^m$, $K_y = K_{y1}z^m$, $K_z = K_{z1}z^m$ coefficients d'échange suivant les axes de coordonnées, K_{x1}, K_{y1}, K_{z1}, y_1 constantes données, m, n nombres rationnels. La concentration de vapeur sur la surface d'eau vérifie les conditions limites: $q = F(x, y)$; q limité à l'infini. Les auteurs donnent la solution effective du problème sous la forme intégrale, comportant les fonctions *V. A. Kostitzen.*

Krzyżański, Miroslaw. Sur la solution élémentaire de l'équation de la chaleur—note complémentaire. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **13**, 24–25 (1952).

The following theorem is a slight generalization of an earlier result of the author [same Rend. (8) **8**, 193–199 (1950); these Rev. **12**, 105] and was communicated to him by D. V. Widder. Both results follow directly from a theorem of Widder [Trans. Amer. Math. Soc. **55**, 85–95 (1944); these Rev. **5**, 203]. If $u(x, y)$ is a regular non-negative solution of $u_y = u_{xx}$ in the region $0 \leqq y \leqq h$, $x^2 + y^2 > 0$ and if $u(x, 0) = 0$, $x \neq 0$, then $u = Ay^{-1/2} \exp\{-x^2/4y\}$. Here A and h are non-negative constants. *F. G. Dressel.*

Madejski, J. Solution of Laplace's equation for certain boundary conditions and its application to the theory of heat conduction in homogeneous isotropic solid bodies. Arch. Méc. Appl., Gdańsk **3**, 147–156 (1951). (Polish. Russian summary).

In this paper there is considered the stationary heat exchange through the walls of a conduit with square cross section and of infinite length. The method is essentially that which N. H. Arutyunyan used for solving the torsion problem of a rod of polygonal cross section [Akad. Nauk SSSR. Prikl Mat. Meh. **13**, 107–112 (1949); these Rev. **10**, 651]. The solution is obtained in the form of an infinite series.

H. P. Thielman (Ames, Iowa).

Ward, G. N. On the integration of some vector differential equations. I. Quart. J. Mech. Appl. Math. **5**, 432–440 (1952).

This paper is concerned with a generalisation of the classical theory of Stokes and Helmholtz which yields a representation of any given field vector in terms of its curl and divergence. Let P and Q be the curl and divergence of two vectors f and g, respectively, and let $g = \psi f$ where ψ is a constant symmetric tensor (dyadic). Let V be a volume bounded by a surface S. Then f and g are represented in V in terms of P and Q within V and in terms of the components of f and g which are parallel and normal to the directed surface elements dS of S, respectively. Elliptic and

hyperbolic cases are treated separately, according to the signature of the quadratic form which corresponds to ψ. The analysis is carried out directly in terms of the vectors concerned, and not in terms of scalar and vector potentials as in the older methods. Use is made of a vector identity which was derived by Ward and Ursell in an earlier paper [Quart. J. Mech. Appl. Math. 3, 326–348 (1950); these Rev. 14, 218].

The equations of electrostatics and of linearised compressible flow are included in the analysis as special cases.

A. Robinson (Toronto, Ont.).

Šapiro, Z. Ya. The first boundary problem for an elliptic system of differential equations. Mat. Sbornik N.S. 28(70), 55–78 (1951). (Russian)

The system considered is of the form

$$\text{(i)} \qquad \sum_{i,k=1}^{3} A_{ik}\frac{\partial^2 u}{\partial x_i \partial x_k}=0$$

where the A_{ik} are constant $n\times n$ matrices and $u=u(x_1, x_2, x_3)$, an unknown n-vector. It is assumed that the form

$$Q(s^1, s^2, s^3)=\det\left(\sum_{i,k=1}^{3} A_{ik}s^i s^k\right)$$

is definite and irreducible. The author proves that for this system the Dirichlet problem in the half-space $x_3>0$ has an (explicitly given) solution for arbitrarily given bounded continuous data $u(x_1, x_2, 0)$. She also reduces the Dirichlet problem for an arbitrary smoothly bounded domain to a system of Fredholm integral equations. This is accomplished by constructing fundamental solutions of (i) and representing the desired solution by an integral analogous to the potential of a double layer. *L. Bers* (New York, N. Y.).

Višik, M. I. On boundary problems for systems of elliptic differential equations and on the stability of their solutions. Doklady Akad. Nauk SSSR (N.S.) 86, 645–648 (1952). (Russian)

Let L be a differential operator

$$Lu=-\sum\frac{\partial}{\partial x_i}\left(A_{ij}(x)\frac{\partial u}{\partial x_j}\right)+\sum B_i(x)\frac{\partial u}{\partial x_i}+C(x)u(x),$$

where u is a function defined in real n-space with values in N-dimensional unitary space (scalar product $(u(x), v(x))$) and all the coefficients are $N\times N$ continuous matrices. The following mixed boundary value problem (I) is considered: (1) $Lu=h$ in a region D with boundary $\Gamma=\Gamma_1\cup\Gamma_2$; (2) $u=\varphi_1$ on Γ_1; (3) $\sum E_{ij}\cos(\nu, x_i)\partial u/\partial x_j+Qu=\varphi_2$ on Γ_2. (Here $E_{ij}+E_{ji}=2A_{ij}$; ν the interior normal; Q a continuous linear operator from $L_p(\Gamma_2)$ to $L_q(\Gamma_2)$, $p<2(n-1)/(n-2)$, $q>2(n-1)/n$.) The operator L is supposed to be elliptic in the sense that the bilinear form

$$E(f, g)=\int_D\sum(E_{ij}(x)\partial f/\partial x_i, \partial g/\partial x_j)dx$$

is supposed to satisfy $\Re E(f, f)+(f, f)\geqq c(\Delta(f, f)+(f, f))$ $(c>0)$, where $\Delta(f, f)$ is Dirichlet's integral and

$$(f, f)=\int_D(f(x), f(x))dx.$$

Then I and the adjoint problem I* constitute a Fredholm pair and when it exists the solution u depends in a certain sense continuously on Q, the coefficients of L and the functions h, φ_1 and φ_2. The corresponding eigenvalue problem has discrete eigenvalues whose real parts are bounded from below. No proofs.

L. Gårding (Lund).

Simonoff, N. Résolution de certains problèmes limites pour les systèmes elliptiques linéaires d'ordre quelconque. Moskov. Gos. Univ. Učenye Zapiski 100, Matematika 1, 53–84 (1946). (Russian. French summary)

The paper considers the boundary value problem in the domain $-\infty<x_i<+\infty$, $i=1, 2, \cdots, n-1$, $x_n>0$ for certain elliptic systems with constant coefficients for p unknown functions $u_k(x_1, \cdots, x_n)$. Linear boundary conditions involving the values of u_k and their derivatives are prescribed along $x_n=0$. The first part of the paper treats the formal solution of this problem which can be reduced, by Fourier transformations, to a system of ordinary differential equations. A more precise analysis is given for the "first" boundary value problem in which the values of the unknown functions and of some of their x_n-derivatives are prescribed on $x_n=0$. This problem is shown to to be correctly set provided that one imposes auxiliary conditions on the given boundary data and on the solutions, conditions requiring a sufficiently fast dying out at infinity. The cases of a single equation and of a system containing only derivatives of even order are considered separately. The paper contains some very formidable formulas and a detailed exposition cannot be given here.

L. Bers (New York, N. Y.).

Lopatinskiĭ, Ya. B. The normal fundamental solutions of a system of linear differential equations of elliptic type. Doklady Akad. Nauk SSSR (N.S.) 78, 865–867 (1951). (Russian)

The system considered is of the form

$$\text{(i)} \qquad \sum_{l=1}^{p}\sum_{j_1+\cdots+j_n\leq s} f_{j_1\cdots j_n}^{kl}\frac{\partial^{j_1+\cdots+j_n}u_l}{\partial x_1^{j_1}\cdots\partial x_n^{j_n}}, \quad k=1, 2, \cdots, p,$$

where the $f_{j_1\cdots j_n}$ are $j_1+\cdots+j_n$ times continuously differentiable functions, and for $(\alpha_1, \cdots, \alpha_n)\neq(0, \cdots, 0)$

$$\det\left(\sum_{j_1+\cdots+j_n=s} f_{j_1\cdots j_n}^{kl}\alpha_1^{j_1}\cdots\alpha_n^{j_n}\right)\neq 0.$$

The system adjoint to (i) is

$$\text{(ii)} \qquad \sum_{k=1}^{p}\sum_{j_1+\cdots+j_n\leq s}(-1)^{j_1+\cdots+j_n}\frac{\partial^{j_1+\cdots+j_n}(f_{j_1\cdots j_n}^{kl}u_k)}{\partial x_1^{j_1}\cdots\partial x_n^{j_n}}=0, \quad l=1, 2, \cdots, p.$$

In a previous note [same Doklady (N.S.) 71, 433–436 (1950); these Rev. 11, 725] the author constructed fundamental matrices $\omega_{kl}(x, y)$ of (i) whose columns are solutions of (i) with singularities at $y=(y_1, \cdots, y_n)$. He calls such a matrix normal if the transposed matrix is a fundamental matrix of (ii). In this note the existence proof for normal fundamental matrices is sketched under the hypotheses that $f_{j_1\cdots j_n}^{kl}$ is $j_1+\cdots+j_n+t$ times continuously differentiable, $t>0$ ($t\geqq 2$ if $j_1+\cdots+j_n=s=1$). The result is used to derive the analyticity of solutions of (i) in the case of analytic coefficients.

L. Bers (New York, N. Y.).

Vol'pert, A. I. The Dirichlet problem in the plane for an elliptic system of linear differential equations of the 2nd order. Doklady Akad. Nauk SSSR (N.S.) 79, 185–187 (1951). (Russian)

The author considers the Dirichlet problem, for a domain D in the z-plane bounded by a Hölder-smooth contour L, and for the elliptic system

$$A^{20}u_{xx}+2A^{11}u_{xy}+A^{02}u_{yy}+A^{10}u_x+A^{01}u_y+A^{00}u=0$$

Here the A^{ij} are $n\times n$ matrices whose elements are $i+j+2$ times continuously differentiable functions of (x, y) and $u=u(x, y)$ is the unknown vector $u=(u_1, \cdots, u_n)$. Using the fundamental solutions constructed by Lopatinskiĭ [same Doklady (N.S.) **71**, 433–436 (1950); these Rev. **11**, 725; see also the preceding review] the author reduces this boundary value problem to a singular integral equation.

L. Bers (New York, N. Y.).

*Gårding, Lars. **Dirichlet's problem and the vibration problem for linear elliptic partial differential equations with constant coefficients.** Proceedings of the Symposium on Spectral Theory and Differential Problems, pp. 291–299. Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1951. $3.00.

Let $p(\xi)=p(\xi_1, \cdots, \xi_n)$ be a real homogeneous polynomial of degree $2m$, $p(\xi)>0$ for $\xi\neq 0$. Solutions $f(x)=f(x_1, \cdots, x_n)$ of (1) $pf(x)=(-1)^m p(\partial/\partial x)f(x)=0$ are sought in a bounded domain S of the n-dimensional Euclidean x-space. The author states that the results are easily extended to the equation (2) $qf(x)=pf(x)+rf(x)=0$, where r is a differential polynomial with constant coefficients of degree $<2m$ such that $r(\xi)=r(-\xi)$, $q(\xi)\geqq 0$ for all ξ.

In order to formulate the boundary condition precisely let $C_\infty(S)$ consist of all infinitely differentiable functions vanishing outside a compact subset of S. Let $|f|_m^2$ denote the sum of the square-integrals of all partial derivatives of $f(x)$, $x\in S$, of order m, and let H denote the closure of $C_\infty(S)$ under the norm $|f|_m$. The problem consists in finding a solution u of (1) such that $u-g\in H$ where g is a given function, $|g|_m<+\infty$.

This problem is solved by the following steps. (a) the fundamental solution $Q(x)$ of (2) is constructed explicitly and shown to be analytic for $x\neq 0$. (b) Using (a) it is shown that every weak solution of (1), that is a function orthogonal to pf, $f\in C_\infty(S)$, is an analytic solution. (c) A metric $|f|$ connected with p is defined in $C_\infty(S)$ and shown to be equivalent to $|f|_m$. This metric can be written as

$$|f|^2=(2\pi)^{-n}\int p(\xi)\,|F(\xi)|^2 d\xi,$$

$F(\xi)$ being the Fourier transform of f. (d) Using these two metrics the existence of u is derived from the fact that every linear functional in Hilbert-space is a scalar product.

At the end of the paper results on the eigenvalue problem are stated (see the following review). *L. Bers.*

Gårding, Lars. The asymptotic distribution of the eigenvalues and eigenfunctions of a general vibration problem. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **21**, no. 11, 10 pp. (1951).

Let $p(\xi)=p(\xi_1, \cdots, \xi_n)$ be a real polynomial of degree m. Let D be a bounded domain in the n-dimensional x-space, and let $\{\lambda_i\}$, $\{\phi_i\}$ be the eigenvalues and eigenfunctions of the equation

$$(1)\qquad p(i\partial/\partial x)f(x)-\lambda f(x)=0$$

the boundary condition being that the derivatives of f of order $\leqq m/2$ should vanish on the boundary of D. This boundary condition is to be understood in the "in the mean" sense: $f(x)$ is to belong to the space H defined in the preceding review. Assuming that $m>n$ and that the eigenfunctions and eigenvalues are normalized and ordered in the usual way, the author proves that as $t\to\infty$

$$(2)\qquad N(t)=\sum_{\lambda_i\leqq t}1=(2\pi)^{-n}t^{n/m}\left\{\int_{\tilde{p}(\xi)\leqq 1}d\xi\right\}\left\{\int_D d\xi\right\}[1+o(1)]$$

and

$$(3)\qquad \lim_{N\to\infty}\left\{N^{-1}\sum_1^N \overline{\phi_k(x)}\phi_k(y)\right\}=\delta_{xy}\left\{\int_D d\xi\right\}^{-1}$$

where $\tilde{p}$ is the principal part of p and $\delta_{xy}=0, 1$ for $x\neq y$, $x=y$. These relations follow, by means of a Tauberian theorem of Hardy and Littlewood, from the asymptotic relation

$$(4)\qquad \lim_{t\to\infty}t^{1-(n/m)}g_t(x, y)=\delta_{xy}(2\pi)^{-n}\int[\tilde{p}(\xi)+t]^{-1}d\xi,$$

where $g_t(x, y)$ is the Green's function of the differential operator $p_t(i\partial/\partial x)$; $p_t=p+t$, for the domain and boundary conditions considered.

Actually a more general problem is considered. Let $p(x)$ be a function such that (B) $p(\xi)=\tilde{p}(\xi)+r(\xi)$, $\tilde{p}(\xi)$ is homogeneous of degree $m>0$, $\inf_{|\xi|=1}\tilde{p}(\xi)>0$, $r(\xi)/\tilde{p}(\xi)\to 0$ as $|\xi|\to\infty$. In the space $C_\infty(D)$ defined in the preceding review define the scalar product

$$p_t(f_1, f_2)=(2\pi)^{-n}\int F_1(\xi)\overline{F_2(\xi)}[p(\xi)+t]d\xi,$$

where $t>0$ and F_i is the Fourier transform of f_i. Closure of $C_\infty(D)$ with respect to the norm $p_t(f, f)^{1/2}$ leads to the Hilbert space H independent of t. In H the formula $\int_D f^2dx=p_t(G_tf, f)$ defines a positive self-adjoint linear operator G_t which the author shows to be completely continuous and an integral operator of the form $G_tf(x)=\int g_t(x, y)f(y)dy$. For the function $g_t(x, y)$ equation (4) holds, and from (4) one obtains (2), (3) where now $\psi_j=\phi_j(\lambda_j+t)^{-1/2}$ are eigenfunctions of G_t orthonormalized by p_t; $G_t\psi_j=\psi_j/(\lambda_j+t)$ for all $t>0$.

The author's method is inspired by Carleman's paper [Åttonde Skandinaviska Matematikerkongressen, Stockholm, 1934, Ohlsson, Lund, 1935, pp. 34–44]. The results generalize those of Weyl, Courant and Pleijel on the equations $\Delta u=\lambda u$, and $\Delta\Delta u=\lambda^n$. *L. Bers.*

Berg, Paul W., and Lax, Peter D. Fourth order operators. Univ. e Politecnico Torino. Rend. Sem. Mat. **11**, 343–358 (1952).

The main result of this paper is the statement that if M and L are linear, elliptic, second order differential operators with identical second order terms, and if D is any open, connected set of n-space with compact closure and boundary B, then there exists a real-valued function u over D satisfying $LMu=f$, u square integrable and "sufficiently" differentiable over D and $u=\text{grad } u=0$ on B (in a certain weak sense defined in terms of L and M), if and only if f is orthogonal to all square integrable functions of the homogeneous adjoint problem defined by M^*L^*. Unfortunately the authors do not formulate precisely any theorems or lemmas, so that it is not clear what they have proved.

F. H. Brownell (Seattle, Wash.).

Malmheden, Harry. The covariant solution of Cauchy's problem for Maxwell's equations in their relativistic form by means of Riesz integrals. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 152–159 (1952).

The object of this paper is to give a covariant solution of Cauchy's problem for Maxwell's equations in their classical relativistic form

$$F_{\alpha\beta,\gamma}+F_{\beta\gamma,\alpha}+F_{\gamma\alpha,\beta}=0,\quad F^{\alpha\beta}{}_{,\beta}=-s^\alpha,$$

where s^α is the four-dimensional current vector, $F_{\alpha\beta}$ the skew-symmetric covariant electromagnetic vector, $F^{\alpha\beta}$ its

contravariant form, in space-time of metric $g_{\lambda\mu}dx^\lambda dx^\mu$; suffixes after a comma denote covariant differentiation. Then $F_{\alpha\beta}$ has to be found in some neighborhood of an open space-like hypersurface on which $F_{\alpha\beta}$ takes given values.

If the vector potential ψ_α is introduced, so that

$$F_{\alpha\beta}=\psi_{\beta,\alpha}-\psi_{\alpha,\beta},$$

the system of equations reduces to

$$\Delta\psi^\alpha+R_\beta{}^\alpha\psi^\beta-\text{grad}^\alpha\,(\text{div}\,\psi)=-s^\alpha,$$

where Δ is the generalised Laplace operator and $R_\beta{}^\alpha$ the contracted curvature tensor. The author shows that it suffices to consider the case div $\psi=0$, and that then methods developed in an earlier paper [same Comm. **8**, 1–116 (1947); these Rev. **9**, 440] lead to the desired solution.

E. T. Copson (St. Andrews).

Protter, M. H. Uniqueness theorems for the Tricomi problem. J. Rational Mech. Anal. **2**, 107–114 (1953).

Tricomi problems are boundary problems involving equations of mixed elliptic-hyperbolic type. The author considers an equation of the form

$$(*)\qquad L(u)\equiv K(y)u_{xx}+u_{yy}=0,$$

where $K(y)$ is a monotone increasing function of y with a continuous second derivative, and $K(0)=0$. This equation is elliptic for $y>0$ and hyperbolic for $y<0$. The (ordinary) differential equations of the partial differential equation (*) are given by

$$(**)\qquad \frac{dy}{dx}=\frac{1}{\sqrt{-K(y)}},\quad \frac{dy}{dx}=\frac{-1}{\sqrt{-K(y)}}.$$

Tricomi's problem (T) is the following: to find a solution of (*) in the domain D (enclosed by the curves Γ_1, Γ_2, Γ_3) assuming prescribed values on Γ_1 and Γ_2. Here Γ_1 is a rectifiable Jordan arc lying in the upper half-plane with endpoints P and Q lying on the positive x-axis. Γ_2 is a characteristic of the first family (**) emanating from P, Γ_3 is a characteristic of the second family (**) emanating from Q.

F. Frankl established the uniqueness of the solution in the case where the function

$$F(y)=2\left(\frac{K}{K'}\right)'+1$$

is positive for all negative values of y [cf. Frankl, Izvestiya Akad. Nauk SSSR. Ser. Mat. **9**, 121–143 (1945); these Rev. **7**, 496]. In the present paper this uniqueness is generalized as follows: Let $K(y)$ be a monotone increasing function with a continuous second derivative, $K(0)=0$, $F(0)>0$, $K'(y)\neq 0$, for $y<0$. Let D be the domain described above. There exists a constant $d_0<0$ such that if $F(y)\geqq d_0$ in D, and $u(x, y)$ is a quasi-regular solution of (*) which vanishes on Γ_1 and Γ_2, then $u\equiv 0$ in D. Let y_m be the maximum of the ordinates of points on Γ_1. Then the author shows the following, where $K(y)$ and D have the same meaning as before: There exists a constant $d_1>0$ such that if $y_m<d_1$ and $u(x, y)$ is a quasi-regular solution of D which vanishes on Γ_1 and Γ_3, then $u\equiv 0$ in D. Frankl investigated the case of a jet flow out of a symmetrical vessel bounded by two straight infinite walls making an angle $2\Theta_0$. He showed that for velocities in the transonic range the problem reduces to that of Tricomi and solved the uniqueness problem under the hypothesis: $F(y)\geqq 0$, $\sigma\leqq 0$, $K(\sigma)\psi_{\sigma\sigma}+\psi_{rr}=0$. For the case under consideration this corresponds to limiting the angle Θ_0 of the jet to 54 degrees. The author shows that as far as uniqueness is concerned the angle 54° is of no physical significance and that uniqueness always holds for angles greater than 54°.

M. Pinl (Dacca).

Frankl', F. I. On a new boundary problem for the equation

$$y\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0.$$

Moskov. Gos. Univ. Učenye Zapiski **152**, Mehanika 3, 99–116 (1951). (Russian)

A new boundary-value problem is considered for the Tricomi equation (*) $yu_{xx}+u_{yy}=0$ in a mixed elliptic-hyperbolic domain. Let Γ_1 be the normal curve $9(x-\frac{1}{3})^2+4y^3=9/4$, $y\geqq 0$, with endpoints $A(0, 0)$ and $B(1, 0)$; let Γ_2 and Γ_3 be the characteristics of (*) which emanate from A and B respectively. Then Γ_1, Γ_2 and Γ_3 enclose a domain D. Define, for $y\leqq 0$, a curve Γ in D with the properties: (i) in a neighborhood of A, Γ coincides with Γ_2; (ii) Γ has negative slope; (iii) if Γ_2' is a characteristic of the same family as Γ_2 and situated "sufficiently close" to Γ_2, then Γ lies completely in the strip formed by Γ_2 and Γ_2'. The curves Γ_1, Γ, and Γ_3 enclose a domain E. The problem considered is the determination of a solution of (*) in E assuming prescribed values on Γ_1 and Γ. By generalizing the method employed by Tricomi [Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (5) **14**, 134–247 (1923)] for the case $\Gamma\equiv\Gamma_2$ the author reduces the problem to the Fredholm alternative. This result combined with (unpublished) uniqueness theorems obtained by C. Morawetz and by the reviewer independently yields the complete existence and uniqueness for the boundary-value problem.

M. H. Protter.

Germain, P., et Bader, R. Sur quelques problèmes relatifs à l'équation de type mixte de Tricomi. O.N.E.R.A. Publ. no. **54**, ii+57 pp. (1952).

Es handelt sich um eine eingehende Darstellung der im Titel genannten und in ausführlicher Weise auch der diesen vorhergehenden Probleme aus der Theorie der Darboux'schen Differentialgleichungen sowie der hyperbolischen und elliptischen singulären Differentialgleichungen.

Im ersten Abschnitt diskutieren die Verfasser Tricomi's Gleichung in der Form

$$u_{xx}+u_{yy}+\frac{1}{3y}u_y=0,\quad \text{bzw.}\quad u_{rr}+\frac{1-t^2}{r}u_{tt}+\frac{4}{3r}\left(u_r-\frac{t}{r}u_t\right)=0.$$

Eine dreiparametrige Transformationsgruppe gestattet unmittelbar zu jeder Lösung $u(x, y)$, bzw. $u(r, t)$, weitere Lösungen $u(x+\mu, y)$, $u(\lambda x, \lambda y)$, $u'(r, t)=r^{-1/3}u(1/r, t)$ zu gewinnen. Zunächst werden dann die geometrischen Eigenschaften dieser Transformationen besprochen (Zusammenhänge mit der Geometrie in Poincaré's Halbebene, usw.) und zwei interessante topologische Modelle der elliptischen und der hyperbolischen Halbebene gewonnen. Damit lässt sich das reguläre Verhalten transformierter Lösungen $u'(P')$ längs uneigentlicher Charakteristiken präzisieren sowie der "reguläre Fortsetzungsprozess der Ordnung n" einer Lösung. Der Ansatz $u=r^{-s-1/6}f_s(t^2)$ führt auf die hypergeometrische Differentialgleichung

$$E\left(\frac{1}{12}+\frac{s}{2},\frac{1}{12}-\frac{s}{2},\frac{2}{3},t^2\right)=0$$

für $f_s(t^2)$ (Methode der Separation der Variablen). Linearkombinationen der beiden Fundamentallösungen von $E=0$ werden als Darboux'sche erste Lösungen bezeichnet und diskutiert. Sie verhalten sich singulär auf den uneigentlichen

Charakteristiken sowie auf den vom Ursprung ausgehenden eigentlichen Charakteristiken. Durch spezielle Transformationen (insbesondere Inversionen) gelangen die Verfasser zu Darboux's allgemeinen Lösungen und zur Riemannschen Funktion des Problems. Im zweiten Abschnitt zerfällt die Untersuchung in zwei Teile, je nachdem ob hyperbolische oder elliptische Probleme vorliegen. Im ersten Falle wird von den vorhergehenden Ergebnissen über die Riemannsche Funktion Gebrauch gemacht, im zweiten Falle wird die entsprechende Greensche Funktion explizit konstruiert und Dirichlet's Problem durch eine zur Poissonschen analoge Formel gelöst. Von besonderer Bedeutung erweisen sich Hadamard's Methoden zur Behandlung der Potentiale einfacher und doppelter Schichten längs parabolischer Kurven. Im dritten Abschnitt wird die Existenz der Lösungen des Tricomischen Problems behandelt, erst im Falle normaler, später im Falle allgemeiner Berandung: es gibt eine und nur eine Lösung $u(x, z)$ der Differentialgleichungen von Tricomi, definiert im gegebenen Gebiet mit stetigen Werten am Rande und zweimal stetig differenzierbaren charakteristischen Werten. Auch das inhomogene Tricomi-Problem wird mit Erfolg behandelt. *M. Pinl* (Dacca).

Douglis, Avron. Some existence theorems for hyperbolic systems of partial differential equations in two independent variables. Comm. Pure Appl. Math. **5**, 119–154 (1952).

In questo interessante lavoro l'autore integra, nel campo reale, il sistema differenziale iperbolico quasi-lineare, nelle incognite $u_j(x, y)$:

$$(1)\quad \sum_{j=1}^{n} p_{ij}(u_{j,x}+z_i u_{j,y})=r_i \qquad \left(i=1, \cdots, n;\ u_{j,x}=\frac{\partial u_j}{\partial x},\ u_{j,y}=\frac{\partial u_j}{\partial y}\right)$$

con le condizioni $u_j(0, y)=\bar{u}_j(y)$. Si suppongono le p_{ij}, z_i, r_i funzioni degli argomenti $x, y, u_1, \cdots, u_n$ e la matrice $(p_{ij}(x, y, u))$ non degenere. L'autore considera dapprima il caso lineare. In tale ipotesi il sistema (1) viene trasformato, mediante integrazioni lungo le caratteristiche, in un sistema di equazioni integrali lineari che si risolvono nell'ipotesi che i coefficienti e i dati abbiano derivate prime continue. Il caso non lineare viene studiato con un procedimento di iterazioni nel quale si utilizza il teorema di esistenza del caso lineare, unitamente a un criterio di compattessa per sistemi non lineari. In tal modo l'esistenza viene dimostrata nelle ipotesi (assai larghe) che i dati $\bar{u}_j(y)$ e i coefficienti abbiano derivate prime continue rispetto agli argomenti $(x, y, u_1, \cdots, u_n)$. Si studia infine il sistema non normale

$$F^i(x, y, u_j, u_{j,x}, u_{j,y})=0$$

riconducendosi al caso precedente mediante derivazioni.

L. Amerio (Milano).

Zwirner, Giuseppe. Sull'equazione

$$\frac{\partial^2 z}{\partial x \partial y}=f\left(x, y, z, \frac{\partial z}{\partial y}\right).$$

Ann. Univ. Ferrara. Sez. VII. (N.S.) **1**, 9–16 (1952).

Two comparison theorems and a uniqueness theorem are given for solutions of this differential equation subject to the conditions $z(a, y)=\psi(y)$ and $z(x, c)=\varphi(x)$.

F. A. Ficken (Knoxville, Tenn.).

Volkov, D. M. Integrals of higher orders of the type of conservation laws for linear hyperbolic problems. Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 255–278 (1951). (Russian)

This paper is a continuation of an earlier paper by the same author [same Izvestiya **15**, 75–90 (1951); these Rev. **13**, 41] to which we refer for definitions. These second order integrals are written out explicitly for the first, second, and third (homogeneous) boundary value problems for the wave equation. The existence of Nth order integrals is proved for all N. Also, certain integrals involving higher order derivatives are obtained for some other boundary-value problems involving the equations of elasticity and Maxwell's equations. *L. Bers* (New York, N. Y.).

Olevskiĭ, M. N. On Riemann's function for the differential equation

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2}+[\rho_1(x)+\rho_2(t)]u=0.$$

Doklady Akad. Nauk SSSR (N.S.) **87**, 337–340 (1952). (Russian)

Let $R_1(x, t; x_0, t_0)$, $R_2(x, t; x_0, t_0)$, $R(x, t; x_0, t_0)$ be the Riemann functions for the equations $u_{xx}-u_{tt}+\rho_1(x)u=0$, $u_{xx}-u_{tt}+\rho_2(t)u=0$, $u_{xx}-u_{tt}+[\rho_1(x)+\rho_2(t)]u=0$, respectively. It is shown that

$$R(x, t; x_0, t_0)=R_1(x, t; x_0, t_0)+\int_{t-t_0}^{x-x_0} R_1(x, \xi; x_0, 0)\frac{\partial R_2(\xi, t; 0, t_0)}{\partial \xi}\,d\xi.$$

The author also proves that R_1 and R_2 are functions of x, x_0, $t-t_0$ and that these functions are symmetric in the variables x, x_0 and even as functions of $t-t_0$. The method of successive approximation is used. *M. H. Protter.*

Pignedoli, Antonio. Sull'aspetto analitico di due importanti problemi della fisica nucleare. Rend. Sem. Mat. Fis. Milano **22** (1951), 74–89 (1952).

The hyperbolic partial differential equation is associated with certain aspects of the theory of nuclear forces, and the parabolic equation inters the theory of scattering of slow neutrons. The present paper discusses known methods of finding solutions of these differential equations meeting various boundary or initial conditions. *F. G. Dressel.*

Sobolev, S. L. On a new problem for systems of partial differential equations. Doklady Akad. Nauk SSSR (N.S.) **81**, 1007–1009 (1951). (Russian)

Sobolev, S. L. Cauchy's problem for a partial case of systems not belonging to the Kowalewsky type. Doklady Akad. Nauk SSSR (N.S.) **82**, 205–208 (1952). (Russian)

These two notes deal with the system

$$\text{(i)}\qquad \frac{\partial \mathbf{v}}{\partial t}-(\mathbf{v}\times\mathbf{k})+\operatorname{grad} p=\mathbf{F},\quad \operatorname{div}\mathbf{v}=\psi,$$

where $\mathbf{v}$ is an unknown vector, p an unknown scalar, $\mathbf{k}$ a fixed unit vector, $\mathbf{F}$ a given vector, ψ a given scalar. This system is not of the Kowalewski type, and is connected with the single 4th order equation

$$\text{(ii)}\qquad \frac{\partial^2 \Delta\Phi}{\partial t^2}+\frac{\partial^2 \Phi}{\partial z^2}=f.$$

In the first note the homogeneous case is considered ($\mathbf{F}=\psi=0$). The solutions are required to exist for $0 \leqq t$ and

for $(x, y, z) \in \Omega$ where Ω is either the whole space (initial value problem) or bounded by a smooth surface S on which either the normal component of $\mathbf{v}$ or p is required to vanish (mixed initial and boundary value problem). By Hilbert space methods these problems are shown to be correctly set.

In the second note the initial value problem for (i) and (ii) is solved explicitly with the aid of certain particular solutions of the homogeneous equation (ii). *L. Bers.*

Myškis, A. D. The simplest boundary problem for generalized systems of telegraph equations. Mat. Sbornik N.S. 31(73), 335–352 (1952). (Russian)

The method of separation of variables was used by N. A. Brazma [Doklady Akad. Nauk SSSR (N.S.) **76**, 41–44 (1951); these Rev. **12**, 657] to solve formally in terms of a series of normal modes the homogeneous vector-matrix telegraphic equation with simple boundary conditions. The present author develops, in continuation of previous work [Mat. Sbornik N.S. **30(72)**, 317–328 (1952); these Rev. **14**, 51], an extensive rigorous theory justifying this expansion, covering also the inhomogeneous case. An important role is played in the investigation by the generalised solutions of S. L. Sobolev [Some applications of functional analysis in mathematical physics, Izdat. Leningrad Gos. Univ., 1950; these Rev. **14**, 566] and by the concept of a "quadratically continuous" function. There are numerous theorems giving conditions on the initial data which ensure the convergence, in mean or uniformly, of the series solution, and various degrees of smoothness of the generalised or ordinary solution. The author proposes, as important unsolved problems, the cases of variable coefficients, or of more than two independent variables. *F. V. Atkinson* (Ibadan).

Pólya, Georges. Sur une interprétation de la méthode des différences finies qui peut fournir des bornes supérieures ou inférieures. C. R. Acad. Sci. Paris **235**, 995–997 (1952).

In many problems involving partial differential equations what is desired is not the full solution of the equation but a certain quantity which can be characterized as the minimum of a variational expression. The author indicates a method by which to obtain a finite difference equation approximating to the partial differential equation in such a way that the value of the desired quantity in the finite difference approximation is definitely an upper bound for the correct value of the quantity. Two applications are given: one, to the torsional rigidity of a bar, and the other to the vibrations of a membrane. The reviewer has constructed the following illustration of the method. Consider the eigenvalues of the equation $u''(x)+\lambda u(x)=0$, $u(0)=u(1)=0$. It is known that $\lambda=\int_0^1 u'^2dx/\int_0^1 u^2dx$. Assume u has arbitrary values u_k at $x_k=k/n$, $k=0, 1, \cdots, n-1$, and assume $u(x)$ is linear between x_k and x_{k+1}. The ratio $\int_0^1 u'^2dx/\int_0^1 u^2dx$ will be a minimum if the u_k satisfy the equations:

$$(u_{k+1}+u_{k-1}-2u_k)/n^2+\lambda(4u_k+u_{k+1}+u_{k-1})/6=0.$$

If $\lambda^*_1 \lambda^*_2, \cdots, \lambda^*_{n-1}$ are the eigenvalues arranged according to size of this set of linear equations, then $\lambda_1 \leqq \lambda^*_1, \lambda_2 \leqq \lambda^*_2, \cdots, \lambda_{n-1} \leqq \lambda^*_{n-1}$, where $\lambda_1, \lambda_2, \cdots, \lambda_{n-1}$ are the eigenvalues arranged in order of magnitude of the differential equation. *B. Friedman* (New York, N. Y.).

Garnir, H. G. Sur une forme générale des distributions résolvantes des opérateurs linéaires à coefficients constants. Bull. Soc. Roy. Sci. Liège **20**, 693–706 (1951).

Formulation et démonstration, en termes de distributions, de résultats obtenus par Gårding [Acta Math. **85**, 1–62 (1951); ces Rev. **12**, 831] et Leray [Cours du Collège de France, 1950–51; Séminaire Bourbaki, mai 1951; Colloque de Bruxelles sur les équations aux dérivées partielles, juin 1951]. Soit $L(\partial/\partial x)$ un opérateur différentiel à coefficients constants dans R^n. Son image de Laplace est un polynome $L(p)=L(\xi+i\eta)$, $p=(p_1, p_2, \cdots, p_n)$. L'ensemble Γ des points ξ pour lesquels $L(\xi+i\eta)$ ne s'annule pas pour η réel a un intérieur dont les composantes connexes sont des ensembles convexes $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$ [Bochner, Amer. J. Math. **59**, 732–738 (1937)]. (1) Pour $\xi \in \Gamma_\alpha$, la fonction $1/L(p)$ est l'image de Laplace d'une distribution E_α [Schwartz, Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 196–206 (1952); ces Rev. **14**, 639], qui est une solution élémentaire de L: $L(\partial/\partial x)*E_\alpha=\delta$. (2) Soit γ_α le cone directeur de Γ_α. Le support de E_α est contenu dans son cone "polaire" c_α, ensemble des $x \in R^n$ tels que $\langle x, \xi\rangle=x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n \geqq 0$ pour tout $\xi \in \gamma_\alpha$ (Leray). On peut même voir que c_α est l'enveloppe convexe fermée de ce support [résultat de Lions: l'article cité de Schwartz et article de Lions à paraître]. (3) Si L est un opérateur d'évolution, on demande une solution élémentaire E de support contenu dans le demi-espace $x_n=t \geqq 0$. Pour que cette méthode fournisse une telle solution, il faut et il suffit que, pour t réel $\geqq 0$ assez grand, $L(i\eta_1, i\eta_2, \cdots, i\eta_n+t)$ ne s'annule pas pour η réel (Gårding).

L'auteur donne des exemples classiques. Certaines démonstrations me semblent insuffisantes (page 699 avec L^*, page 701 juste avant §6). *L. Schwartz* (Paris).

Functional Analysis, Ergodic Theory

***Fréchet, Maurice. Pages choisies d'analyse générale.** Gauthier-Villars, Paris; E. Nauwelaerts, Louvain, 1953. 213 pp. 2000 francs; $5.94.

This volume contains selections by the author from his papers on general analysis. He has also supplemented the original papers with short remarks on later developments connected with the ideas in them, answers to conjectures, etc. The papers are grouped under the following headings: Vue d'ensemble; Espaces fonctionnels; Analyse fonctionnelle; Les espaces abstraits; L'analyse générale.

Sebastião e Silva, J. Sui fondamenti della teoria dei funzionali analitici. Portugaliae Math. **12**, 1–47 (1953).

In this paper the author continues his study of the theory of analytic functionals [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 709–715 (1946); Portugaliae Math. **9**, 1–130 (1950); Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **1**, 23–102 (1950); these Rev. **8**, 278; **11**, 524; **12**, 715]. In many respects the present paper is devoted to amending and supplementing the author's previous work in the light of the work of C. L. da Silva Dias [São Paolo thesis, 1951; these Rev. **13**, 249], some as yet unpublished work of G. Köthe on duality in function theory, and the work of Dieudonné and Schwartz on the theory of duality in linear topological spaces [Ann. Inst. Fourier Grenoble **1**, 61–101 (1950); these Rev. **12**, 417]. The topics dealt with include the following: topological issues in spaces of functions, analytic continuation of functional operators, continuity of operators which are analytic in the sense of Fantappié, convergence theorems for linear operators, and various other considerations about linear operators which map one space of analytic functions into another such space.

Some of these last considerations merely open the way for future studies. The emphasis throughout is on the value of the modern abstract point of view. *A. E. Taylor.*

Ganapathy Iyer, V. On the space of integral functions. III. Proc. Amer. Math. Soc. **3**, 874–883 (1952).

Γ is the $\mathfrak{F}$ space of entire functions under the topology of uniform convergence on compact sets. Previous papers by the author have discussed the dual space and related questions [J. Indian Math. Soc. (N.S.) **12**, 13–30 (1948); Quart. J. Math., Oxford Ser. (2) **1**, 86–96 (1950); these Rev. **10**, 380; **12**, 108]. Given a sequence $\alpha_n \in \Gamma$, the author uses a familiar device of Banach to discuss conditions under which the subspace spanned by the α_n is dense in Γ. For $\alpha_n(z)=\alpha(zz_n)$ the problem reduces to the classical one of uniqueness sets, and a sufficient condition is that $\limsup n/|z_n|^p=\infty$, where α is of order p and finite type. [For a wider class of conditions, see R. P. Boas, Ann. of Math. (2) **47**, 21–32 (1946); these Rev. **7**, 425.] Let $a_n(z)=\sum_0^\infty (a_k)^n z^k$ where $\mathrm{Re}\,(a_k)$ is strictly decreasing. Then, $\{\alpha_1, \alpha_2, \cdots\}$ is fundamental in Γ. Under certain conditions on the sequence $\{z_n\}$ and $\alpha(z)$, it is shown that the closed subspace spanned by the sets of translates $\alpha(z+z_n)$ and the set of derivatives $\alpha^{(k)}(z)$ are the same. He concludes with some remarks about continuous transformations of Γ into itself, and their connections with bases for Γ. It should be remarked that Theorem 5 is a well-known fact about continuous endomorphisms of any convex linear space.

R. C. Buck (Madison, Wis.).

Lorch, E. R. A curvature study of convex bodies in Banach spaces. Ann. Mat. Pura Appl. (4) **34**, 105–112 (1953).

The author studies the effect of differentiability of a function related to the Minkowski functional of a convex body in a Banach space B; for simplicity of statement he takes the body to be the unit sphere and, for fixed $r>1$, sets $G(x)=\|x\|^r/r$. For each fixed $x, y, z \in B$ the first and second partial derivatives of $\varphi(\alpha,\beta)=G(x+\alpha y+\beta z)$ at $\alpha=0=\beta$ will be assumed to exist and will be denoted by $G_y(x)$, $G_{yz}(x)$, etc.

In §II it is assumed that if $x\neq 0$ and $y\neq 0$, then $G_{yy}(x)>0$, a condition which implies strict convexity of B. It is shown that the mapping $x\to G_x(x)$ is one-one from B into B^* and $\|G_x(x)\|=\|x\|^{r-1}$, that an $f \in B^*$ is of the form $tG_x(x)$ if and only if f assumes its maximum on the unit sphere at $x/\|x\|$ and $t=\|f\|\|x\|^{r-1}$, and that, for each $x\neq 0$, $G_{yz}(x)$ is a bilinear function in y, z, and the mapping $y\to G_y(x)$ is a bounded, linear, one-one map of B into B^*.

In §III it is supposed that there exists $L>0$ such that $L\|x\|^{r-2}\|y\|^2 \leq G_{yy}(x)$ for all $x, y \in B$. Then B is uniformly convex, so $x\to G_x(x)$ is onto B^*. Then $G_x(x)\to x$ is one-one and strongly continuous from B^* onto B. Also $y\to G_y(x)$ takes B bicontinuously onto B^* for each $x\neq 0$. If it is assumed that also there exists $K>0$ such that $G_{yy}(x)\leq K\|x\|^{r-2}\|y\|^2$, then $|G_{yz}(x)|\leq K\|x\|^{r-2}\|y\|\|z\|$, the mapping $x\to G_x(x)$ becomes strongly continuous, and, for each choice of $x\neq 0$, B is isomorphic to a Hilbert space with inner product given by $G_{yz}(x)$. *M. M. Day* (Urbana, Ill.).

Gál, I. S. The principle of condensation of singularities. Duke Math. J. **20**, 27–35 (1953).

For a sequence of homogeneous bounded (not necessarily additive) operators $u_n(x)$ on a Banach space E, the author has defined the notion of asymptotic subadditivity and has proved the principle of uniform boundedness [Ann. Inst. Fourier Grenoble **3**, 23–30 (1952); these Rev. **14**, 288]. Now he considers double sequences of operators such that the $u_{mn}(x)$, $n=1, 2, \cdots$, are asymptotically subadditive. Then the existence of $x_m \in E$ with $\limsup_{n\to\infty}\|u_{mn}(x_m)\|=+\infty$, $m=1, 2, \cdots$, implies the existence of a $\xi \in E$ with

$$\limsup_{n\to\infty}\|u_{mn}(\xi)\|=+\infty$$

for each m. This ξ is given in the form $\xi=\sum a_{mn}x_{mn}$, where $\|x_{mn}\|=1$, $\sum|a_{mn}|<+\infty$ and x_{mn} "is in resonance with u_{mn}", that is, $\|u_{mn}\|$ and $\|u_{mn}(x_{mn})\|$ are of the same order. The author says that he was not able to prove a variation of this theorem dealing with divergence instead of unbounded divergence. [A reference to a paper by Kantorovitch is wrong and should be Mat. Sbornik **41**, 503–510 (1934).]

G. G. Lorentz (Toronto, Ont.).

Rothe, E. H. Gradient mappings. Bull. Amer. Math. Soc. **59**, 5–19 (1953).

The paper is of an expository nature, summarising the work on gradient mappings contained in a series of papers by the author [these Rev. **8**, 158; **10**, 461, 548; **11**, 184; **12**, 720; **13**, 254; **14**, 185]. Let $I(x)$ be a real-valued function defined in a Banach space E (or some subset of E), with a continuous Fréchet differential $D(x, h)$; for fixed x, $D(x, h)=l_x(h)$ defines a linear functional l_x on E, i.e., an element of the dual space of E. The mapping $x\to l_x$ of E into its dual is said to be a gradient mapping, and we write $l_x=\mathrm{grad}\,I(x)$. Examples of gradient mappings are provided by (a) bounded symmetric linear operators in Hilbert space, (b) the Euler differential expression $d(f_{x'})/dt-f_x$ of the calculus of variations, under appropriate conditions on $f(t, x(t), x'(t))$, and (c) the mapping taking $x(s)$ into $y(s)+\int K(s, t)f(t, y(t))dt$, where $y(s)=\int K(s, t)x(s)ds$, under appropriate conditions on $K(s, t)$ and $f(t, y(t))$.

The principal applications of gradient mappings arise from the fact that critical points of the scalar function $I(x)$ in the sphere $\{x: \|x\|\leq R\}$ satisfy equations of the form

$$\mathrm{grad}\,I(x)+\lambda x=0. \tag{1}$$

If, for instance, $I(x)$ is weakly continuous, the existence of critical points can be established directly, and existence theorems for solutions of (1) can be obtained. The author describes applications to non-linear differential and integral equations; among the auxiliary tools used are the Leray-Schauder index and the type numbers of Marston Morse.

F. Smithies (Cambridge, England).

San Juan, R. Generalization of a theorem of Steinhaus on linear functionals. Las Ciencias. Madrid **17**, no. 2, 205–208 (1952). (Spanish)

Let U be the space of functions $x(t)$ on $0\leq t\leq 1$, Lebesgue integrable for every interval $0\leq t\leq u<1$, for which $\lim_{u\to 1-}\int_0^u x(t)dt$ exists. This is a linear normed space if $\|x\|=\max_u|\int_0^u x(t)dt|$. The article shows that if $f(x)$ is any linear continuous form on U, then there exists a measurable function essentially bounded on (0, 1), such that $f(x)=\lim_{u\to 1-}\int_0^u x(t)\alpha(t)dt$, convergent for all x of U. A complete characterization of such functions α is not given.

T. H. Hildebrandt (Ann Arbor, Mich.).

Citlanadze, È. S. On a class of nonlinear functional equations. Soobščeniya Akad. Nauk Gruzin. SSR **11**, 73–80 (1950). (Russian)

The author gives, with amplifications and proofs, results stated in a previously reviewed article [Doklady Akad. Nauk SSSR (N.S.) **71**, 441–444 (1950); these Rev. **11**, 670].

If $f(x)$ is a (nonlinear) functional defined on all of L^p and Fréchet differentiable, write

$$f(x+h)-f(x)=(Lx, h)+\omega_f(x, h),$$

where L is a mapping of L^p onto its conjugate space. If x is developed in a series of Haar orthogonal functions, then, if f is weakly continuous, the values of f and L for the nth partial sums converge uniformly inside the unit sphere to the values of f and L for x, the latter provided $\|\omega_f(x; h)\| \leq C\|h\|^2$ for some constant C. Under this condition for ω, weak continuity of f implies complete continuity of L. Complete continuity of L implies that f is weakly continuous and that $f(x)=f(0)+\int_0^1(x, Ltx)dt$. Proper vectors of L are studied; the results are similar to those in the article mentioned.

J. L. B. Cooper (Cardiff).

Šmul'yan, Yu. L. On unconditionally convergent and absolutely convergent series. Uspehi Matem. Nauk (N.S.) 7, no. 6(52), 209–210 (1952). (Russian)

This note contains a standard example showing the inequivalence in l_p spaces of the modes of convergence mentioned in the title. The paper by Dvoretzky and Rogers [Proc. Nat. Acad. Sci. U. S. A. 36, 192–197 (1950); these Rev. 11, 525] is not mentioned in the bibliography.

G. K. Kalisch (Minneapolis, Minn.).

***Følner, E. Sur quelques questions rattachées à deux espaces de dimensions infinies.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 225–228. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

A brief résumé of certain results of the author and of the author and H. Bohr. The results in question have all now been published elsewhere and reviews of the relevant papers have appeared [these Rev. 10, 507; 11, 640, 659; 12, 509].

G. W. Mackey (Cambridge, Mass.).

Nevanlinna, Rolf. Erweiterung der Theorie des Hilbertschen Raumes. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 160–168 (1952).

By considering a not necessarily positive definite Hermitian inner product $Q(x, y)$ in a vector space, the author succeeds in generalizing the notions of orthogonalization, the construction of a spanning "orthonormal" system from a given linearly independent system, etc. To facilitate the discussion (and with an eye to eventual application), axioms concerning $Q(x, y)$ are given which serve the function of those normally associated with Hilbert space. The main application which the author foresees is the use of a generalized Dirichlet principle technique for partial differential equations problems which are not of elliptic type. For these problems, the associated integral is not definite and thus stationary rather than minimum values are obtained. But by the use of the techniques developed, this stationary value is found to be the difference of two minima.

In the following review a more detailed sketch of the theory will be found.

B. R. Gelbaum.

Nevanlinna, Rolf. Über metrische lineare Räume. III. Theorie der Orthogonalsysteme. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 115, 27 pp. (1952).

[For part II see same Ann. no. 113 (1952); these Rev. 14, 287.] Let $Q(x, y)$ be a not necessarily definite Hermitian form in a vector space R and let $a_1, a_2, \cdots$, be linearly independent vectors in R. One may "orthonormalize" any finite set $a_1, \cdots, a_n$ and obtain a new set $x_1, \cdots, x_n$ for which $|Q(x_i, x_j)|=0$ or δ_{ij} and such that the linear sets $L(a_1, \cdots, a_n)$ and $L(x_1, \cdots, x_n)$ are identical. Let $Q(x, x)=Q(x)$. Let r_n be the number of x_i for which $Q(x_i)=+1$, s_n the number for which $Q(x_i)=-1$, t_n the number for which $Q(x_i)=0$. Then $r_n+s_n+t_n=n$, and $r_m+s_m \leqq r_{m+1}+s_{m+1}$. If not all $t_n=n$, a modification of the above method yields a sequence of spaces $L_n=L(x_1, \cdots, x_n)$ for which $|Q(x_i, x_j)|=\delta_{ij}$. The spaces L_n are decomposed into L_n^1 and L_n^2 where L_n^1 (L_n^2) is spanned by the elements x_i for which $Q(x_i)=1$ (-1). For each n, and arbitrary x in R, p_n is defined by $p_n=\sum_{i=1}^{n}Q(x_i)Q(x, x_i)$. The first axiom is now: A: $\sum_{i=1}^{\infty}|Q(x, x_i)|^2<\infty$. The completeness of the system a_1, a_2 is characterized by axiom B: $\lim_{n\to\infty} Q(p_n^2) \leqq Q(x) \leqq \lim_{n\to\infty} Q(p_n^1)$ where p_n^1 (p_n^2) is the projection of x on L_n^1 (L_n^2). Three new forms are introduced:

$$Q_1(x, y)=\sum_{Q(x_i)>0} Q(x, x_i)\overline{Q(y, x_i)},$$

$$Q_2(x, y)=-\sum_{Q(x_i)<0} Q(x, x_i)\overline{Q(y, x_i)},$$

$$P(x, y)=Q_1(x, y)+Q_2(x, y).$$

Then $|Q(x)| \leqq P(x)$ and $P(x, y)$ is used for a (possibly degenerate) metric. With respect to P the Cauchy completeness axiom, C, is imposed. To make $P(x, y)$ nondegenerate, axiom D: For no x, is $Q(x, y)\equiv 0$, is imposed. Then the metric $P(x, y)$ makes R a Hilbert space.

Theorem: The necessary and sufficient condition that a system $a_1, a_2, \cdots$, with the properties described in axioms A and B exists is the existence of a definite majorant form $H(x, y)$ for which (for some constant M)

$$|Q(x, y)| \leqq MH(x)H(y)$$

and for which the Hilbert space with the form $H(x, y)$ is separable. The proof uses the spectral theorem applied to the operator T defined by $Q(x, y)=H(x, Ty)$. The paper concludes with a discussion of minimal majorant forms.

B. R. Gelbaum (Minneapolis, Minn.).

Sard, Arthur. Approximation and variance. Trans. Amer. Math. Soc. 73, 428–446 (1952).

The author considers the Hilbert space $\mathfrak{C}=L_2(E_1, m)$, m being a fixed Lebesgue-Stieltjes measure on the real line E_1, a fixed subspace $\mathfrak{M} \subseteq \mathfrak{C}$, and a fixed stochastic process error $e(t, w) \in L_2(E_1 \times \Omega, m \times p)$, (Ω, p) being a probability measure space, such that at each t the expected value $\mathrm{ex}[e(t, \cdot)]=0$. He defines for $\mathfrak{M}$ an approximation operator as a bounded linear operator A on $\mathfrak{C}$ having $\mathrm{ex}[Af+Ae]=f$ for $f \in \mathfrak{M}$, or thus equivalently $(A-I)\mathfrak{M}=0$, such that $\mathrm{ex}[\|Ae\|^2]$ is minimal over such A. He also considers a bounded linear self-adjoint operator W on $\mathfrak{C}$ having $(Wf, f) \geqq 0$ for $f \in \mathfrak{C}$ and $\|PWf\| \geqq b\|f\|$ for $f \in \mathfrak{M}$ for some fixed $b>0$, where P is the orthogonal projection onto $\mathfrak{M}$. For such W, called a weight, a least square operator for $\mathfrak{M}$ is defined as a bounded linear operator C on $\mathfrak{C}$ such that $C\mathfrak{C} \subseteq \mathfrak{M}$ and such that $(W(f-Cf), f-Cf)$ is minimal over such C for each $f \in \mathfrak{C}$. Each weight determines a unique least square operator for $\mathfrak{M}$, essentially the projection relative to W onto $\mathfrak{M}$.

The author's main results are: (theorem 1) a bounded A having $(A-I)\mathfrak{M}=0$ is an approximation operator if and only if $VA^*\mathfrak{C} \subseteq \mathfrak{M}$; (theorem 4) the least square operator determined by a weight W is an approximation operator if and only if $VW\mathfrak{M} \subseteq \mathfrak{M}$. This last greatly generalizes the standard statistics result [Aitken, Proc. Roy. Soc. Edinburgh. Sect. A. 62, 138–146 (1945); these Rev. 8, 54] that the least square operator is an approximation operator if

V^{-1} exists, $W=V^{-1}$, and $\mathfrak{X}$ is finite-dimensional. In these statements V is the variance operator, bounded on $\mathfrak{X}$, determined for the error process e by the equation

$$[V(f)](t)=\int_\Omega\left\{\int_{-\infty}^{\infty}f(\tau)\overline{e(\tau,w)}dm(\tau)\right\}e(t,w)dp(w).$$

There are some other subsidiary results, including the definition of curve fitting operators similar to approximation operators if $\mathfrak{M}$ is finite-dimensional. *F. H. Brownell.*

Davis, Chandler. Estimating eigenvalues. Proc. Amer. Math. Soc. **3**, 942–947 (1952).

Let A be a self-adjoint operator in a Hilbert space, $P(\lambda)=\prod_{i=1}^{2n}(\lambda-\alpha_i)$ for $2n$ real numbers α_i,

$$Q_n(\alpha_1,\alpha_2,\cdots,\alpha_{2n})=(P(A)x,x).$$

The following generalisation of a method of Kato [J. Phys. Soc. Japan **4**, 334–339 (1949); these Rev. **12**, 447] is proposed. If the intervals $(\alpha_{2i-1},\alpha_{2i})$ do not overlap the spectrum of A, then Q_n is positive if it exists: if it is known that (α,β) $(\alpha_{2k}<\alpha<\beta<\alpha_{2k+1})$ intersects the spectrum only in one isolated point λ, then if an x be found for which the coefficient of α_{2k} in Q_n is negative, the value of α_{2k} for which this function vanishes gives, in some cases, an improved estimate for the isolated point. If A is bounded, and b is such that there is a single isolated point of the spectrum of A whose distance from b attains the minimum value for the spectrum, then a method for finding this value and a corresponding eigenvector follows from the application of an iterative method of Karush [Proc. Amer. Math. Soc. **2**, 980–989 (1951); these Rev. **13**, 563] to the corresponding problem for the operator $(A-b)^2$. *J. L. B. Cooper.*

Mautner, F. I. On eigenfunction expansions. Proc. Nat. Acad. Sci. U. S. A. **39**, 49–53 (1953).

The author discusses sufficient conditions for the existence of an eigenfunction expansion for a spectral family in a complex Hilbert space of functions. *A. F. Ruston.*

Devinatz, A. Integral representations of positive definite functions. Trans. Amer. Math. Soc. **74**, 56–77 (1953).

Let J be a Hilbert space of functions $f(x)$ defined on a set E, $K(x,y)$ a function on $E\times E$. $K(x,y)$ is called a positive matrix if $\sum a_i\bar{a}_jK(y_i,y_j)\geq 0$ for every finite sum, and, following Aronszajn [same Trans. **68**, 337–404 (1950); these Rev. **14**, 479], is a reproducing kernel (r. k.) for J if $f(y)=(f(x),K(x,y))$ for all $y\in E$, $f\in J$. The author extends to unbounded operators on J results of Aronszajn concerning the correspondence between operators and kernels $M(x,y)$ on $E\times E$. If $E=S\times G$ where S is a semi-group and G a group, $f(x)=f(x_1,x_2)$ is called a positive-definite function if $K(x,y)=f(x_1y_1x_2y_2^{-1})$ is a positive matrix.

If $A(x,y)=\int_{-\infty}^{\infty}a(x,\lambda)\overline{a(y,\lambda)}dV(\lambda)$, where V is a bounded increasing function on $(-\infty,\infty)$, and $a(x,\lambda)\in L_2(V)$ for each x, it is shown that the r.k. space corresponding to the positive matrix $A(x,y)$ consists of all functions of the form $f(x)=\int a(x,\lambda)\varphi(\lambda)dV(\lambda)$ where $\varphi\in L_2(V)$.

The main theorem of the paper gives a representation theorem for a class of positive definite functions. Let $x\in E^m$, $y\in E^n$ the n- and m-dimensional vector spaces: let $z=(x,iy)\in E^n\times E^m$. Necessary and sufficient conditions for a function $f(z)$ defined in a domain $x_r>c_r$ to be of the form $f(z)=\int\exp(z\cdot t)dV(t)$, where the integral is taken over an interval in E^{m+n}, V gives a positive measure in that interval, and $z\cdot t$ denotes the scalar product, are given: the principal condition is that $F(x,y)=f(z)$ should be positive definite. An application to a generalization of the Hausdorff moment theorem is given. *J. L. B. Cooper* (Cardiff).

Ferretti, Bruno. Su di una classe di equazioni operatoriali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 473–476 (1952).

The author proves the existence of a solution of the equation $dA/d\lambda=F(A,\lambda)$ where λ is a real parameter and A is an operator of Hilbert space. The proof is very simple, following the proof of the classical existence theorem for ordinary differential equations. The following assumptions are made concerning A and F. (i) A is a variable Hermitian operator with a purely discrete eigenvalue spectrum. (ii) For each A and λ, F is a bounded Hermitian operator with a purely discrete spectrum. (iii) F is integrable with respect to λ for each A. (iv) For each λ, F satisfies a Lipschitz condition $\|F(A_1,\lambda)-F(A_2,\lambda)\|\leq K\|A_1-A_2\|$. This existence theorem has been used by the author in his papers on quantum field theory [Nuovo Cimento (9) **8**, 108–131 (1951); these Rev. **12**, 890]. *F. J. Dyson* (Ithaca, N. Y.).

Caccioppoli, Renato. Equazioni differenziali ordinarie negli spazi astratti; osservazioni su una nota del prof. B. Ferretti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **13**, 95–97 (1952).

The author criticizes and clarifies the note of Ferretti reviewed above. He points out that Ferretti's theorem is incorrect as stated, because the property of having a discrete spectrum is not a linear property of an operator. For example, let $F_1=x_1x_2+x_3x_4+\cdots$, $F_2=x_2x_3+x_4x_5+\cdots$ be two quadratic forms representing Hermitian operators with purely discrete spectra. Then $F(\lambda)=\alpha(\lambda)F_1+\beta(\lambda)F_2$ has the same property for every λ, when α and β are two real functions satisfying $\alpha(\lambda)\beta(\lambda)=0$, $\int_0^1\alpha(\lambda)d\lambda=\int_0^1\beta(\lambda)d\lambda=1$. However, $\int_0^1F(\lambda)d\lambda=F_1+F_2$ is then an operator with a purely continuous spectrum.

The theorem of Ferretti is correct if the condition of having a discrete spectrum is simply dropped. The author points out the theorem in fact remains true, and can be proved in the same way, under much more general conditions. *F. J. Dyson* (Ithaca, N. Y.).

Putnam, C. R. Function space: Hilbert-space correspondences in quantum mechanics. Quart. J. Math., Oxford Ser. (2) **3**, 260–267 (1952).

The unitary transformation from $L_2(-\infty,\infty)$ to a sequence Hilbert space determined by the choice of the Hermite functions as an orthonormal basis takes the operation $Q: f(x)\to xf(x)$, with domain all $f(x)$ in $L_2(-\infty,\infty)$ for which $xf(x)$ is in $L_2(-\infty,\infty)$, into the operation Q': $\xi=(\xi_1,\xi_2,\cdots)\to(\eta_1,\eta_2,\cdots)$, where

$$\eta_i=\sum_j\alpha_{ij}\xi_j,\quad \alpha_{ij}=2^{-1/2}(j^{1/2}\delta_{i+1,j}+j^{1/2}\delta_{j+1,i}),$$

with domain consisting of all vectors ξ for which $\sum_i|\eta_i|^2<\infty$. *I. E. Segal* (Chicago, Ill.).

Dye, H. A. The unitary structure in finite rings of operators. Duke Math. J. **20**, 55–69 (1953).

Let M be a weakly closed self-adjoint algebra of bounded operators (containing the identity) on a complex Hilbert space H (called a W^*-algebra). Theorem 1 asserts that if M has no minimal projections, then the weak closure of the group M_U of unitary operators $\in M$ is the entire unit sphere of M. This is used to discuss the problem to what extent M

is characterized by M_U. Thus Theorems 2 and 3 assert that under certain conditions an isomorphism of M_U can be extended to M. *F. I. Mautner* (Baltimore, Md.).

Fuglede, Bent, and Kadison, Richard V. Determinant theory in finite factors. Ann. of Math. (2) **55**, 520–530 (1952).

The authors give now the details and proofs of their results on the introduction of determinants in rings of operators of finite type as outlined in Proc. Nat. Acad. Sci. U. S. A. **37**, 425–431 (1951); these Rev. **13**, 255.

F. I. Mautner (Baltimore, Md.).

Dixmier, J. Algèbres quasi-unitaires. Comment. Math. Helv. **26**, 275–322 (1952).

A unimodular group, or more generally a group of measure-preserving transformations on a measure space, gives rise to a ring of operators on a Hilbert space in a fashion first examined, in special cases, by Murray and von Neumann. This ring is conveniently investigated through an alternative Hilbert space construct, a Hilbert space with a distinguished conjugation and a partial multiplication, called here a unitary algebra. However, the rings which arise from non-unimodular groups, or from groups of measurable transformations on a space admitting no equivalent invariant measure, cannot be treated in this way and the present paper develops a generalized construct, a "quasi-unitary algebra", suitable for dealing with these more general cases. This algebra is defined as one admitting an involutory anti-automorphism s and an automorphism j, and being a pre-Hilbert space A with respect to an inner product with the properties: 1) $(x, x^j) \geqq 0$; 2) $(x, x) = (x^s, x^s)$; 3) $(xy, z) = (y, x^{sj}z)$; 4) the map L_x, $y \to xy$, is continuous for fixed x; 5) the set of all $(xy)+(xy)^j$ is dense in A. The extensions of the L_x to the completion of A generate a ring L, the right multiplications generate a corresponding ring R, and it is shown that just as in the case of a unitary algebra, $L' = R$, and that the conjugation that extends s leaves invariant self-adjoint elements of $L \cap R$.

Under various further hypotheses, which, however, are all satisfied in the case of the algebra arising from a group of measurable transformations, results extending the work of Murray and von Neumann on the type problem for this kind of algebra [see esp. von Neumann, Ann. of Math. (2) **41**, 94–161 (1940); these Rev. **1**, 146] are obtained: necessary and sufficient conditions (modulo the appropriate further hypotheses) are given that (a) L have no part of type III, and b) L be of finite type. A central decomposition for separable quasi-unitary algebras is also given.

I. E. Segal (Chicago, Ill.).

Dixmier, J. Remarques sur les applications $\natural$. Arch. Math. **3**, 290–297 (1952).

This paper completes earlier work of the author [Ann. Sci. Ecole Norm. Sup. (3) **66**, 209–261 (1949); Compositio Math. **10**, 1–55 (1952); these Rev. **11**, 370; **14**, 481] with regard to certain points. If ϕ is a normal trace on the ideal I in the ring of operators M, then for any fixed A in I, the map $B \to \phi(AB)$ is weakly and strongly continuous on the unit sphere of M, and is completely additive on projections, in the strong topology. If p and q are positive real with $p^{-1}+q^{-1}=1$, if A and B are in M, and if $(A^*A)^{p/2}$ and $(B^*B)^{q/2}$ are in I, then AB and BA are also in I and $\phi(AB) = \phi(BA)$. *I. E. Segal* (Chicago, Ill.).

Smiley, M. F. Right H^*-algebras. Proc. Amer. Math. Soc. **4**, 1–4 (1953).

A right H^*-algebra is a Banach algebra A whose normed linear space is a Hilbert space and such that for each x in A there is an x^* in A for which right multiplication by x is the adjoint operator to right multiplication by x^*. If one makes the further assumption that left multiplication by x is the adjoint operator to left multiplication by x^*, A is called an H^*-algebra. The purpose of this paper is to extend to right H^*-algebras the known structure theorems for H^*-algebras, in case A is proper in the sense: $Ax=0$ implies $x=0$. The main theorem says that such an A is an H^*-algebra in which the norm has been changed to an equivalent norm in each simple component. *W. Ambrose.*

Rickart, C. E. On spectral permanence for certain Banach algebras. Proc. Amer. Math. Soc. **4**, 191–196 (1953).

If A is a Banach algebra, let ϕ_A be the set of algebraic homomorphisms (into the complex numbers) with the weak topology. A is called regular in case for each closed $F \subset \phi_A$ and each homomorphism $\phi_0 \in \phi_A - F$, there is a $u \in A$ for which $u(F) = \text{constant} \neq u(\phi_0)$. Theorem 1. If A is a semi-simple regular Banach algebra which is algebraically embedded in a Banach algebra B, then every $\phi \in \phi_A$ can be extended to an element of ϕ_B. An element x of A is called quasi-regular in case there is a y in A for which $x+y=xy=yx$. Theorem 2. If A and B are related as above, and if A is a B^*-algebra, then x is quasi-regular in A if and only if x and x^* are quasi-regular in B. Let $\sigma_A(x)$ be the image of x under the homomorphisms of A and let $r_A(x) = \sup |\lambda|$, $\lambda \in \sigma_A(x)$. Corollaries of the above are: 1) For the situation of Theorem 1, $\sigma_A(x) = \sigma_B(x)$, $r_A(x) = r_B(x)$; 2) for the situation of Theorem 2, $\sigma_A(x) = \sigma_B(x) \cup (\sigma_B(x^*)$ and if $xx^* = x^*x$, then $\sigma_A(x) = \sigma_B(x)$. *B. Gelbaum* (Minneapolis, Minn.).

Gel'fand, I. M., and Fomin, S. V. Geodesic flows on manifolds of constant negative curvature. Uspehi Matem. Nauk (N.S.) **7**, no. 1(47), 118–137 (1952). (Russian)

This paper contains for the most part detailed proofs, extensions and generalizations of results announced previously by the authors [Doklady Akad. Nauk SSSR (N.S.) **76**, 771–774 (1951); these Rev. **13**, 473]. Thus the main results of the paper concern spectra of geodesic flows on manifolds of constant negative curvature. The following theorem is proved in detail: A geodesic flow in an arbitrary n-dimensional manifold ($n=2, 3$) of constant negative curvature has a Lebesgue spectrum (i.e., all of its spectral measures are equivalent to the ordinary Lebesgue measure). The possible extension of the above theorem to the case of an arbitrary positive integer n is indicated.

The method of proof for the case $n=2$ is to represent the flow on the surface as the coset space G/D of the group G of all real 2×2 matrices of determinant 1 with respect to a suitable discrete subgroup D. The flow is defined by multiplication by $g_t = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix}$.

In the case $n=3$, the manifold is represented as the totality of two-sided cosets $h = DgK$ ($g \in G$) of the group G of all complex unimodular matrices of order 2 with respect to a suitable discrete subgroup D and a compact subgroup K. The flow is again defined by multiplication by g_t. Thus the point DgK moves with time t into the point Dgg_tK. In both cases the authors appeal to the complete classification of all unitary representations of the group G [V. Bargmann,

Ann. of Math. (2) **48**, 568–640 (1947); these Rev. **9**, 133; I. M. Gelfand and M. A. Naimark, Izvestiya Akad. Nauk SSSR. Ser. Mat. **11**, 411–504 (1947); these Rev. **9**, 495]. They show that in each of these representations the spectrum is a Lebesgue spectrum. Their results follow as a consequence.

From the above result follow the well-known theorems of Hopf and Hedlund on the metric transitivity and mixing properties of geodesic flows in 2- and 3-dimensional manifolds of constant negative curvature and finite volume. In the case $n=2$ the authors also prove by appealing to the study of automorphic forms that the spectrum of the flow is an enumerably multiple Lebesgue spectrum. In the last paragraph the authors discuss the group-theoretical scheme which lies at the basis of their proofs and its applications to coset spaces of some locally compact Lie groups.

Y. N. Dowker (London).

Calculus of Variations

***Weinstock, Robert. Calculus of variations with applications to physics and engineering.** McGraw-Hill Book Company Inc., New York-Toronto-London, 1952. x+326 pp.

Der Verfasser kennzeichnet seine Absicht im Vorwort als eine doppelte: Studenten in die Handhabung der Variationsrechnung einzuführen und die Anwendungen der Variationsrechnung an Beispielen, die nicht im Bereich der reinen Mathematik liegen zu erläutern. Es wird nur die Theorie der ersten Variation besprochen. Die weiteren notwendigen und hinreichenden Bedingungen kommen nicht vor. Die Hamilton-Jacobische partielle Differentialgleichung wird nur im Zusammenhang mit mechanischen Problemen erläutert, aber ihr Nutzen für die Integration der Eulerschen Gleichung wird nicht besprochen. Der Lagrangesche Multiplikator wird nur für gewöhnliche isoperimetrische Probleme behandelt. Bei der Behandlung der Eigenwertprobleme wird das Entwicklungstheorem ohne Beweis postuliert. Die Anwendungen beziehen sich auf das Fermatsche Problem, auf Elastizitätstheorie, Quantenmechanik und Elektrostatik. Bei der Theorie der schwingenden Membran wird im Anschluss an Courant die asymptotische Verteilung der Eigenwerte besprochen. Die Ritzsche Methode wird dargelegt und an Beispielen erläutert.

P. Funk (Wien).

***Damkoehler, Guillermo. Definiteness and reversibility in the calculus of variations.** Symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Diciembre, 1951, pp. 41–63. Centro de Cooperación Científica de la Unesco para América Latina, Montevideo, Uruguay, 1952. (Spanish)

The paper consists of interesting comments of an elementary nature on regular variational problems in the large for simple integrals in parametric form in the plane, and these lead up to a number of particular results to be published in detail elsewhere. The author uses ideas of his early work [Ann. Scuola Norm. Super. Pisa (2) **5**, 127–140 (1936)] to avoid not only the assumption of definiteness, but also that of reversibility which sometimes replaces definiteness. The results announced include the following: Given a positively regular variational problem for curves situated on a sufficiently smooth surface S of the type of the sphere, there exists a simple closed extremal of minimax type and index 0 or 1. There is a corresponding result on a torus.

L. C. Young (Madison, Wis.).

Flodin, Bertil. Über eine Art stetiger Lösungen bei Variationsproblemen mit Gefällbeschränkung. Soc. Sci. Fenn. Comment. Phys.-Math. **15**, no. 20, 14 pp. (1951).

This paper continues the analysis of the problem of minimizing an integral

$$I=\int_{x_1}^{x_2} f(x, y, y')dx$$

where admissible curves lie in the (x, y)-plane and their slopes are subject to the restrictions $a\leqq y'\leqq b$. [Cf. the earlier paper by the author in Acta Soc. Sci. Fennicae. Nova Ser. A. **3**, no. 10 (1945); these Rev. **7**, 208.] The case here considered is that where the minimizing curve is composed of a straight line segment C_1 with slope a or b, and a segment C_2 of an extremal tangent to C_1. Two examples are examined: the first is Newton's problem of the solid of revolution of minimum resistance with restrictions on the slope of the generating curve; the second is that in which $f(x, y, y')=(1-y'^2)^{n+1/2}$, $n\geqq 3$.

L. M. Graves.

Mancill, J. D. One-sided maxima and minima of functions of two or more variables. Amer. Math. Monthly **60**, 80–88 (1953).

The author treats necessary conditions and sufficient conditions for an extreme of a function $f(x, y, z)$ at a point P which is not a critical point but lies on the boundary S of the region of admissible points. These conditions are expressed in terms of the higher derivatives of the values taken by f along S. For the case of two independent variables, the case when P is at a corner of S is also treated. The author notes that the main results extend at once to functions of n variables. The problem in n dimensions when there is more than one constraint was treated by W. Karush [M. S. Dissertation, Univ. of Chicago, 1939] and as a preliminary lemma by L. L. Pennisi [Doctoral Dissertation, Univ. of Chicago, 1952, Sec. 3; see also the paper reviewed below] for the case where it is not necessary to consider derivatives of order higher than the second. The present paper has a preliminary section treating free extremes of functions of two variables by means of polar coordinates, giving conditions in terms of derivatives of higher order. A case overlapping this one (the second variation vanishes but not identically) was treated by M. Friedman [Doctoral Dissertation, Univ. of Chicago, 1952].

L. M. Graves (Chicago, Ill.).

Pennisi, Louis L. An indirect sufficiency proof for the problem of Lagrange with differential inequalities as added side conditions. Trans. Amer. Math. Soc. **74**, 177–198 (1953).

The present paper is concerned with obtaining a sufficiency theorem for a strong relative minimum for the problem of minimizing an integral

$$I(C)=\int_{x^1}^{x^2} f(x, y, \dot{y})dx$$

in a class of arcs C: $y_i(x)$ $(i=1, \cdots, n;\ x^1\leqq x\leqq x^2)$ joining two fixed points and satisfying a set of differential inequalities $\varphi^\beta(x, y, \dot{y})\geqq 0$ and a set of differential equations

$\psi^\rho(x, y, \dot{y}) = 0$. The sufficiency theorem obtained is a very general one, and is the best that has been obtained to date. It affords an excellent example of the greater effectiveness of indirect methods over methods based on Mayer fields in establishing sufficiency theorems. The analogous problem for functions of a finite number of variables is discussed also. *M. R. Hestenes* (Los Angeles, Calif.).

Cooperman, Philip. The multiplier rule for ordinary differential equations. Proc. Amer. Math. Soc. **3**, 143–146 (1952).

In variational problems involving differential side conditions the proof of the multiplier rule involves two steps. The first step is to show that a minimizing arc is zero for all admissible variations satisfying variational differential side conditions. The second step is to show that the results given in the first step yields the multiplier rule. The first step is the more difficult one. Unless normality assumptions are made, the proof is a complicated one [see M. J. Cox, Amer. J. Math. **66**, 170–198 (1944); these Rev. **6**, 129]. In the present paper the author obtains conditions which are equivalent to the multiplier rule. However, he does not show that a minimizing arc must satisfy these conditions. Moreover, the multiplier rule as stated by the author has been established under the tacit assumption that the problem is normal. *M. R. Hestenes* (Los Angeles, Calif.).

Karush, William. Isoperimetric problems in the calculus of variations. Canadian J. Math. **4**, 257–280 (1952).

This paper is concerned with giving a direct proof of the usual sufficiency theorem of simple integrals of parametric type in a class of curves with variable end points and satisfying isoperimetric side conditions. Other proofs in the literature are indirect proofs or expansion theorems or involve transformation of the problem to one of Bolza type. The proof here given employs an extension of the concepts of Mayer fields and makes use of a generalization of Hahn's lemma. *M. R. Hestenes* (Los Angeles, Calif.).

Karush, William. The index of an extremal arc. Canadian J. Math. **4**, 281–294 (1952).

This paper is concerned with several definitions of the index of an extremal of a simple integral of parametric type subject to variable endpoint conditions. Four definitions are given and conditions for their equivalence is discussed. The first of these involves the index of the second variation, the second involves families of curves, the third is expressed in terms of isoperimetric problems, and the fourth involves topological considerations. *M. R. Hestenes.*

Glansdorff, P. Sur une forme nouvelle en cascade du terme aux limites de la variation d'une intégrale multiple. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 136–153 (1952).

The present paper is concerned with the form of the first variation of a multiple integral. Following methods used by Th. De Donder the author expresses the first variation as sums of integrals of order $0, 1, \cdots, n$ where n denotes the number of independent variables. There is no restriction on the number of independent variables nor on the order of the derivatives appearing in the integrand. *M. R. Hestenes.*

Lewy, Hans. On minimal surfaces with partially free boundary. Comm. Pure Appl. Math. **4**, 1–13 (1951).

This paper is in a certain sense a continuation of the author's paper on the continuation of minimal surfaces across analytic boundary arcs [H. Lewy, Proc. Nat. Acad. Sci. U. S. A. **37**, 103–110 (1951); these Rev. **14**, 168] but deals with an even more difficult situation. Let S be a closed surface in 3-space and γ a rectifiable Jordan arc joining two distinct points on S. It is assumed that the endpoints of γ can be joined by a rectifiable curve on S, and that no inner points of γ lie on S. Let K denote the class of all surfaces ν defined by a continuous mapping of the semi-disc D: $u^2+v^2 \leq 1$, $v>0$, into 3-space, such that the arc $u^2+v^2=1$ is mapped topologically onto γ and the distance from the image of (u, v) to S approaches zero if $v \to 0$. Courant proved [cf. R. Courant, Dirichlet's principle, conformal mapping and minimal surfaces, Interscience, N. Y., 1950, Chapter VI; these Rev. **12**, 90] that among all surfaces in K there is one (minimal) surface of least area, μ. Let γ' denote the complement to γ in the boundary of μ, the boundary being the set of limits of sequences of points of μ corresponding to sequences of parameter points (u, v) tending to the boundary of D. In 1945 Courant [Proc. Nat. Acad. Sci. U. S. A. **31**, 242–246 (1945); these Rev. **7**, 68] discussed the character of γ' for convex S. In this paper the author shows, assuming only that S is a closed orientable surface analytically embedded in 3-space, that γ' is an analytic curve of finite length joining the endpoints of γ. *L. Bers.*

Theory of Probability

Lipschutz, Miriam. Generalization of a theorem of Chung and Feller. Proc. Amer. Math. Soc. **3**, 659–670 (1952).

Let $X_1, X_2, \cdots$ be independent random variables with a common distribution, such that $E(X_k)=0$ and $E(X_k^2)=1$. Let $S_n=X_1+\cdots+X_n$, and let N_n denote the number of those terms in the sequence $S_1, S_2, \cdots, S_n$ which either are positive, or which are zero but follow a positive term. For the particular case when the common X_k distribution has only the possible values ± 1, each with the probability $\frac{1}{2}$, Chung and Feller [Proc. Nat. Acad. Sci. U. S. A. **35**, 605–608 (1949); these Rev. **11**, 444] studied the conditional distribution of N_{2n} under the hypothesis $S_{2n}=0$, and proved that under this hypothesis N_{2n} assumes each of its possible values $0, 2, \cdots, 2n$ with the probability $1/(n+1)$. In the present paper, a corresponding result is proved for the more general case when the X_k may assume any integer values, $E(X_k^4)$ being finite. Under the hypothesis $S_n=0$, the variable N_n/n will then be asymptotically (as $n \to \infty$) distributed over the interval $(0, 1)$. *H. Cramér* (Stockholm).

Mourier, Edith. Eléments aléatoires laplaciens dans un espace de Banach. C. R. Acad. Sci. Paris **236**, 575–576 (1953).

A random variable with values in a Banach space is laplacian if every linear functional on the space is a (numerical) laplacian random variable. Several theorems are stated under the hypothesis that the Banach space is a separable Hilbert space. For example, in this case, if X is a laplacian random variable, $E\{\|X\|^2\} < \infty$; if $\{x_k\}$ is a complete orthonormal sequence in the space, $X=\sum A_k x_k$, where the A_k's are laplacian and are mutually independent if $\{x_k\}$ is chosen properly. *J. L. Doob* (Urbana, Ill.).

Fourgeaud, Claude. Probabilité conditionnelle et régression linéaire dans les espaces de Banach. C. R. Acad. Sci. Paris **236**, 576–578 (1953).

A Banach space point-valued random variable is said to have linear regression relative to a second if the correspond-

ing fact is true for all pairs of linear functionals on the two spaces. A condition for this is found in terms of the characteristic functional of the given pair. *J. L. Doob.*

***Karhunen, Kari. Über ein Extrapolationsproblem in dem Hilbertschen Raum.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 35–41. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

The author considers a stochastic process

$$\{x(t), -\infty < t < \infty\}$$

with stationary (wide sense) increments. Kolmogorov [C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 6–9 (1940); these Rev. **2**, 220] has found the general form of the covariance function for such a process. The usual prediction theory concepts of indeterministic and deterministic processes are applied to the $x(t)$ processes. Such a process is indeterministic [deterministic] if and only if the stationary $x(t)-x(t-h)$ process has this property for every value of h. Analytic conditions for these properties are found, as well as simple representations of the processes. *J. L. Doob.*

***Mann, Henry B. Introduction to the theory of stochastic processes depending on a continuous parameter.** National Bureau of Standards Applied Mathematics Series, no. 24. U. S. Government Printing Office, Washington, D. C. vi+45 pp. $.30.

L'auteur réussit la gageure d'enclore en 44 pages un exposé d'ensemble sur les processus stochastiques, correct et dense, où les points de vue physiques sont bien saisis; il est vrai qu'il se limite aux processus permanents qui sont des fonctions numériques (à 1 dimension) et aux propriétés déterminées uniquement par la loi temporelle; et l'exposé est insuffisant en ce qui concerne les processus de Poisson et les processus de Markoff, dont la théorie proprement dite n'est pas abordée; l'énoncé du corollaire du Lemme 1.7, p. 6, n'est pas tout à fait correct.

L'auteur rappelle les points fondamentaux sur la convergence stochastique des variables aléatoires, sur l'analyse stochastique des fonctions aléatoires (en particulier du second ordre, et en fonction de la covariance), puis étudie la fonction de Wiener-Levy du mouvement brownien classique (qu'il appelle le processus aléatoire fondamental), et le processus dérivé de Ornstein-Uhlenbeck; il aborde les problèmes statistiques concernant ces deux types de processus (problèmes d'estimation; les travaux de Grenander ne sont pas rappelés).

L'auteur donne des indications sur les fonctions aléatoires à accroissements indépendants en général, les fonctions aléatoires dérivées d'une fonction à accroissements indépendants, les problèmes concernant les compteurs Geiger (travaux de W. Feller), et termine sur l'analyse harmonique des fonctions de Wiener-Levy et des fonctions stationnaires du second ordre, et les lois des grands nombres concernant ces dernières. *R. Fortet* (Paris).

Onoyama, Takuji. Random frequency process. Bull. Math. Statist. **5**, 51–58 (1952).

The author considers a simple type of stationary stochastic process, evaluating the spectral representation explicitly. *J. L. Doob* (Urbana, Ill.).

Lévy, Paul. Complément à l'étude des processus de Markoff. Ann. Sci. Ecole Norm. Sup. (3) **69**, 203–212 (1952).

This is a complement to the author's recent paper on the same subject [same Ann. (3) **68**, 327–381 (1951); these Rev. **13**, 959]. For definitions and notations see the cited review. Processes of types $\leqq 5$, especially type 5, are considered. Let A_0 be an instantaneous state and let $\tau(t)=m(E_0\cap(0,t))$. A main idea is to use τ as a new time parameter. Let A_1 be another state and $\tau'(t)=m(E_1\cap(0,t))$. Then $\tau'/\tau\to\lambda_{01}/\lambda_{10}$ with probability one. Here λ_{01} is given as follows: starting from A_0, the value of τ corresponding to the first realization of A_1 has the probability density $\lambda_{01}e^{-\lambda_{01}u}$. This generalizes the author's previous result (Theorem I.3.2) for a discrete time process. The process $X(\tau)=t-\tau$, introduced in the previous paper, is studied in more detail. Under the tacit assumption that A_0 is the only instantaneous state, $X(\tau)$ is of independent stationary increments, increasing only by jumps. Let $E[e^{iuX(\tau)}]=\tau\psi(z)=\tau\int_0^\infty(e^{iux}-1)F(x)dx$. It is shown how to form $\psi(t)$ in several stages, depending on the complicity of the possible "types" of the jumps. The author says that $\psi(z)$ is far from sufficient to determine the process. He conjectures that $\psi(z)=\sum\psi_k(z)$ where $\psi_k(z)$ is an entire function of $(z+\lambda_k i)^{-1}$, etc. However, some $-\lambda_k i$ may not be a singularity of $\psi(z)$, as shown by a simple example. *K. L. Chung* (Ithaca, N. Y.).

Taylor, S. J. The Hausdorff α-dimensional measure of Brownian paths in n-space. Proc. Cambridge Philos. Soc. **49**, 31–39 (1953).

It is proved that the Hausdorff dimension number of almost all Brownian paths in n-space ($n\geqq 2$) is 2; and that almost all these paths have zero Hausdorff 2-measure. This is a weaker result than announced by P. Lévy [C. R. Acad. Sci. Paris **233**, 600–602 (1951); these Rev. **13**, 363]. The quantity $\mu\{\omega: \sup_{0\leq t\leq 1}|X(t,\omega)|<x\}$ is known exactly [Lévy, Processus stochastiques et mouvement brownien, Gauthier-Villars, Paris, 1948, Theorem 42.1; these Rev. **10**, 551]. Lemma 7 and Corollary, essential for the proof of the second statement, are actually implicit in a paper by the reviewer [Trans. Amer. Math. Soc. **64**, 205–233 (1948); these Rev. **10**, 132], according to which we have indeed $\liminf_{t\to 0}\sup|X(t,\omega)|/[t(\log\log t^{-1})^{-1}]^{1/2}=\pi/8^{1/2}$ for almost all ω. The transference from sums of independent random variables to the local theorem for the Brownian movement process is straightforward. In fact, it is simpler to prove the Brownian movement version because all estimates become exact [cf. Khintchine, Asymptotishe Gesetze der Wahrscheinlichkeitsrechnung, Springer, Berlin, 1933, Chapter V]. *K. L. Chung* (Ithaca, N. Y.).

Foster, F. G. A Markov chain derivation of discrete distributions. Ann. Math. Statistics **23**, 624–627 (1952).

The author gives a simple rule for constructing a transition matrix of an irreducible aperiodic Markov chain such that its nth powers tend to an arbitrarily selected "distribution matrix" (with identical rows). *M. Loève.*

Montroll, Elliott W. Markoff chains, Wiener integrals, and quantum theory. Comm. Pure Appl. Math. **5**, 415–453 (1952).

An expository article devoted to a detailed discussion of connections between the mathematical formalism of quantum mechanics and that of random functions. While the exposition does not pretend to be rigorous, the point of view and the comparisons made should prove to be instructive not only for theoretical physicists but also for probabilists. *M. Loève* (Berkeley, Calif.).

Hermans, J. J., and Ullman, R. The statistics of stiff chains, with applications to light scattering. Physica 18, 951–971 (1952).

The authors investigate "worm-like" chains, where the length of each step approaches zero. The method is that applied by Kramers to Brownian motion [Physica 7, 284–304 (1940); these Rev. 2, 140]. The results are essentially those of Daniels [Proc. Roy. Soc. Edinburgh. Sect. A. 63, 290–311 (1952); these Rev. 14, 295]. But, contrary to Daniels, the authors do not eliminate at the very beginning the orientation of the chain, hence can obtain sometimes supplementary information. *M. Loève.*

***Jensen, Arne. Distribution patterns, composed of a limited number of exponential distributions.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 210–215. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

The author considers Markov chains of the discontinuous type, with a finite number of states, and whose transition probabilities satisfy

$$P'_{ij}(t_0, t) = \sum_{\nu=0}^{n} P_{i\nu}(t_0, t) a_{\nu j}$$

where $a_{ij} \geqq 0, i \neq j$, $a_{ij} \leqq 0$ for $i = j$ and $\sum a_{ij} \leqq 0$. For a state i with $\sum a_{ij} < 0$, $P_{iE}(t_0, t) < 1$ for $t > t_0$ and hence the group of states E is capable of decreasing. This decrease is associated with an event A having intensity $A_{j,n+1}$ and distribution given by $p_{iA}(t_0, t)dt = \sum_{j=0}^{n} P_{ij}(t_0, t) A_{j,n+1} dt$ with

$$0 \leqq A_{i,n+1} \leqq -\sum a_{ij}, \quad i = 1, \cdots, n.$$

If $A_{j,n+1} = -\sum a_{jk}$ for all j then $\int_{t_0}^{\infty} p_{iA}(t_0, t)dt = 1$. The author gives examples of birth type processes with $a_{ij} = 0$ for $j > i+1$, $a_{\nu\nu} = -(\alpha + \nu\beta)$, $\nu = 0, \cdots, n$. Then $P_{ij}(t_0, t)$ is a sum of exponentials and $p_{nA}(t_0, t)$ is, after a time transformation, of Pearson type. *J. L. Snell.*

Riordan, John. Delay curves for calls served at random. Bell System Tech. J. 32, 100–119 (1953).

The paper presents curves and tables for the distribution (not probability as written by the author) of delay of calls served by a simple trunk group with assignment of delayed calls to the trunks at random and with pure chance call input, the holding times having an exponential distribution. The results are contrasted with the classic results of A. K. Erlang (the Danish mathematician, not engineer, as often thought). The author refers for proof and tables not to the original work of Erlang, but to later works of Molina, and seems not to know the tabulation of the original C function by A. Jensen [Moe's principle . . . , Copenhagen, 1950; these Rev. 12, 624]. The notations used are not convenient for readers without knowledge of the different traditions in the different telephone administrations.

As the distribution of delay is a special case of the distribution given by A. Jensen [see the preceding review], it is possible to calculate easily the tables with any desired accuracy. The accuracy of the author's figures are not given. The method of indirect use of the moments of the distribution and the representation by some exponential terms have not resulted in a concrete discussion of the accuracy obtained.

A. Jensen (Copenhagen).

Mathematical Statistics

***Neyman, Jerzy. Lectures and conferences on mathematical statistics and probability.** 2nd. ed. Graduate School, U. S. Department of Agriculture, Washington, D. C., 1952. viii+274 pp.

Cet ouvrage est une seconde édition, assez fortement remaniée, d'un texte miméographié paru en 1938; le fond de la plupart des chapitres est tiré de conférences prononcées par l'auteur à la Graduate School, U. S. Department of Agriculture, et s'attache par conséquent davantage aux idées qu'aux détails de calcul ou même qu'aux démonstrations. Voici les principaux points traités:

Dans une partie d'introduction, l'auteur pose sa définition de la probabilité (selon laquelle, essentiellement, une probabilité est une fréquence), rappelle quelques points élémentaires nécessaires du Calcul des Probabilités, développe quelques considérations sur l'applicabilité du Calcul des Probabilités a un problème concret, puis aborde le problème de tester une hypothèse statistique, à propos duquel il rappelle d'abord quelques procédés anciens dont il montre l'insuffisance; puis, partant des notions d'erreurs de première espèce et de deuxième espèce, il expose le principe de la méthode qu'il a tant contribué à rendre classique (puissance d'un test, tests sans erreur systématique, etc.) et l'illustre d'un exemple emprunté à la génétique [à propos de cet exemple: n'aurait-il pas fallu signaler que si, pour la simplicité, il est naturel de tester à partir de x_1 et x_2 seulement, n_2 (cf. p. 59) contient cependant de l'information quant au choix entre H_1 et H_2; et, p. 62, pourquoi ne pas indiquer que $R(e_k)$ est un résumé exhaustif].

Le chapitre II discute, sur des exemples précis développés, quelques "questions controversées" concernant l'expérimentation agricole (dans le cas d'arrangements réguliers peut-on appliquer les formules relatives aux arrangements aléatoires? etc.); le chapitre III traite de l'échantillonnage, en particulier stratifié, des populations humaines; la seconde partie de ce chapitre en particulier étudie, en fonction du budget disponible pour procéder à l'échantillonnage le très intéressant problème suivant: le caractère à étudier C étant corrélé avec un second caractère C_1, peut-il être avantageux de déterminer d'abord la répartition de C_1, pour procéder ensuite à l'égard de C à un échantillonnage stratifié en fonction de C_1. La troisième partie du même chapitre traite des corrélations introduites artificiellement.

Le quatrième et dernier chapitre étudie l'estimation des paramètres; l'auteur expose la théorie classique de Bayes, puis la méthode des intervalles de confiance, en mettant très clairement en évidence la différence des deux points de vue; puis, en examinant s'il existe quelque parenté entre la méthode des intervalles de confiance et la notion de "probabilité fiduciaire" de R. A. Fisher, il procède à une critique, vraisemblablement définitive, de celle-ci. *R. Fortet.*

Gumbel, E. J., Greenwood, J. Arthur, and Durand, David. The circular normal distribution: theory and tables. J. Amer. Statist. Assoc. 48, 131–152 (1953).

The authors investigate the properties of the circular normal probability function

$$f(\alpha) = \frac{\exp k \cos(\alpha - \alpha_0)}{2\pi I_0(k)}$$

and indicate its use in various fields. The maximum likelihood estimates of α_0 and k are derived. In table 2 are listed the solutions of $\bar{a} = I_1(k)/I_0(k)$ to 3 decimals, $0 \leqq k \leqq 4.18$, where $\bar{a}$ is a function of the sample values. In table 3 are

listed the radii of the probability function for equiareal tracing of $\psi(\alpha)=[(\exp k\cos\alpha)/I_0(k)]^{1/2}$ when $0\leqq\alpha\leqq180°$, $k=0(.1)4.0$ to three decimal places. In table 4 areas are given for

$$\Phi(\alpha)=\frac{1}{2\pi}\int_{-\alpha}^{\alpha}\frac{\exp k\cos x}{I_0(k)}dx$$

to 5 decimals when $\alpha=0°(5°)180°$, $k=0(.2)4$.

L. A. Aroian (Culver City, Calif.).

Sagrista, Sebastian Navarro. On a generalization of Pearson's curves to the two-dimensional case. Trabajos Estadística **3**, 273–314 (1952). (Spanish. English summary)

The author makes a thorough study of the statistical features, moments, marginal distributions, and regression of the bivariate probability functions

$$(1)\qquad z=k\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{2rxy}{ab}\right)^n,\quad n+1>0,\ r^2<1;$$

and

$$(2)\qquad z=k(h^2+ax^2+2bxy+cy^2)^n\exp-\tfrac{1}{2}(ax^2+2bxy+cy^2).$$

Apparently the author is not familiar with the papers of van Uven who provides a complete treatment of (1) [Nederl. Akad. Wetensch., Proc. **50**, 1063–1070, 1252–1264 (1947); **51**, 41–52, 191–196 (1948); these Rev. **9**, 363, 452] and R. Risser [Assoc. Actuair. Belges. Bull. no. **53**, 5–48 (1948); these Rev. **10**, 312]. *L. A. Aroian* (Culver City, Calif.).

Masuyama, Motosaburo. The misclassification in the sampling inspection. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. **1**, no. 4, 7–9 (1952).

N observations are taken, with replacement, from a multinomial population with probabilities $p_1,\cdots,p_r$. An observation of type i has probability a_{ij} of being reported as type j; $i,\ j=1,\cdots,r$. If the matrix (a_{ij}) is known and is nonsingular, the maximum likelihood estimate of p_i is $\hat{p}_i=\sum_j x_jA_{ji}/N$, where x_j is the number reported as type j, and $(A_{ij})=(a_{ij})^{-1}$. The covariance of $\hat{p}_i$ and $\hat{p}_j$ is found.

T. E. Harris (Santa Monica, Calif.).

David, F. N., and Johnson, N. L. The truncated Poisson. Biometrics **8**, 275–285 (1952).

The authors are concerned with the estimation of the parameter λ in the truncated Poisson probability function when the observation in the zero class is missing. They first discuss the problem of the goodness of fit using the index of dispersion. If the probability function is representable by a Poisson, they derive and compare the maximum likelihood estimate of λ and what is essentially a moment estimate of λ. In case the Poisson probability function is not suitable, similar methods are given for the negative binomial series $(q-p)^{-s}$. A related problem has been considered by P. G. Moore [Biometrika **39**, 247–251 (1952); these Rev. **14**, 391]. *L. A. Aroian* (Culver City, Calif.).

Sandelius, Martin. Truncated inverse binomial sampling. Skand. Aktuarietidskr. **34**, 41–44 (1951).

From an infinite population consisting of a's and b's in proportions p and $1-p$, respectively, individuals are drawn at random until one of the two events occurs: 1) a given number M of a's are drawn, 2) a preassigned sample-size $N(>M)$ is reached. For this procedure an unbiased estimate of p is exhibited and the variance of this estimate is obtained.

Z. W. Birnbaum (Seattle, Wash.).

Rushton, S. On a two-sided sequential t-test. Biometrika **39**, 302–308 (1952).

Discussion for the two-sided alternative exactly like that of Barnard [Biometrika **39**, 144–150 (1952); these Rev. **14**, 65] for the one-sided alternative to the hypothesis that the mean of a normal population with unknown variance is zero, again resulting in an invariant test differing from Wald's [Sequential analysis, Wiley, New York, 1947; these Rev. **8**, 593] by an increase of unity in the degrees of freedom associated with the t-variables the ratio of whose densities equals the probability ratio criterion. Analogous treatment of the sequential analysis of variances for two groups, including a useful approximation to the test criterion. *J. Kiefer* (Ithaca, N. Y.).

Rosenblatt, M. Limit theorems associated with variants of the von Mises statistic. Ann. Math. Statistics **23**, 617–623 (1952).

For the case of a multi-dimensional rectangular distribution, an analogue of the ω^2-statistic is given. The asymptotic distribution of this statistic is found in a manner indicated by Kac, who used this method to solve the one-dimensional case [Proc. Second Berkeley Symposium on Math. Statistics and Probability, 1950, Univ. of California Press, 1951, pp. 189–215; these Rev. **13**, 568]. For the one-dimensional case of testing whether two samples come from populations with the same continuous c.d.f., the statistic

$$\frac{mn}{2(m+n)}\int[S_1(t)-S_2(t)]^2d[S_1(t)+S_2(t)],$$

where S_1 and S_2 are sample c.d.f.'s from samples of size m and n and where $m/n\to\lambda>0$ as $n\to\infty$, is shown when the null hypothesis is true to have the same limiting distribution as the one-dimensional ω^2-statistic. *J. Kiefer.*

Bennett, B. M. Estimation of means on the basis of preliminary tests of significance. Ann. Inst. Statist. Math., Tokyo **4**, 31–43 (1952).

It is desired to estimate the mean μ of a population. A second population has mean ν which may or may not equal μ. A sample is obtained from each population. The hypothesis $\mu=\nu$ is tested. If the hypothesis is accepted, μ is estimated by a procedure which assumes $\mu=\nu$. If the hypothesis is rejected, μ is estimated by a procedure involving only the sample from the first population. The properties of this type of estimate are investigated in the cases of normal populations with known variances, and with unknown but equal variances. The case of unknown and possibly unequal variances involves two preliminary tests.

H. Chernoff (Stanford, Calif.).

Lyerly, Samuel B. The average Spearman rank correlation coefficient. Psychometrika **17**, 421–428 (1952).

If N individuals each rank n objects and if it is desired to find the average of the N Spearman rank correlation coefficients, ρ, obtainable on computing ρ between each ranking and a criterion or correct set of ranks, it will be convenient to be able to compute this average directly from the Nn ranks assigned instead of first calculating the N separate ρ's. The author develops a formula for this purpose. He then discusses the use of the average ρ to test the null hypothesis that the expected value of ρ is zero for each of the N rankers. He adduces evidence to show that for all but the small values of N and n (for which the exact distribution on the null hypothesis is calculable as he shows) the use of the

normal approximation will be satisfactory at 5% and 1% significance levels. *C. C. Craig* (Ann Arbor, Mich.).

Miyasawa, Kôichi. Minimax estimations. Bull. Math. Statist. **5**, 59–76 (1952).

The author proves several of the results of Wolfowitz on sequential point and interval estimation of the mean of a normal distribution [Ann. Math. Statistics **21**, 218–230 (1950); these Rev. **12**, 36]. The author considers also one weight function slightly different from those of Wolfowitz, but the latter's proof obviously applies also there, and is in fact used throughout the paper. This involves proving first in great detail the fact that if one stops the first time there is no (a posteriori) profitable continuation and makes the best (a posteriori) terminal decision, one has a Bayes solution. *J. Kiefer* (Ithaca, N. Y.).

Basu, D. On the minimax approach to the problem of estimation. Proc. Nat. Inst. Sci. India **18**, 287–299 (1952).

Mainly an exposition of a few well-known results on non-sequential point estimation. Particular attention is paid to the Poisson case with quadratic weight function and bounded mean; the result of pp. 140–142 of Wald's book [Statistical decision functions, Wiley, New York, 1950; these Rev. **12**, 193], which is easily seen to hold for all distributions of exponential type, is carried out for the Poisson case. The author mistakenly gives an example where every estimator is minimax to illustrate the possibility that a minimax estimator may not exist. *J. Kiefer.*

Lehmann, E. L. Testing multiparameter hypotheses. Ann. Math. Statistics **23**, 541–552 (1952).

Suppose that the distribution of some random variables depends upon s real parameters $\theta_1, \cdots, \theta_s$. Consider the hypothesis H: $\theta_i \leqq \theta_i^*$, $i=1, \cdots, s$. The author proves that under certain regularity assumptions no non-trivial unbiased tests of H exist. Optimum non-randomized minimax tests of several types are derived where the regions of rejection satisfy certain monotonicity conditions. Several examples are given and various related multi-decision problems and two-sided hypotheses are discussed.

R. Peterson (Seattle, Wash.).

Bartlett, M. S. The statistical significance of odd bits of information. Biometrika **39**, 228–237 (1952).

The idea is suggested and explored that if a null hypothesis H attaches the probability $p(x_i)$ to the points x_i of a discrete sample space, then H can be tested by whether $-\log p(x_i)$ lies sufficiently close to its expected value under H. *L. J. Savage* (Chicago, Ill.).

van der Waerden, B. L. Order tests for the two-sample problem and their power. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 453–458 (1952); Corrigenda **15**, 80 (1953).

The power of Wilcoxon's test is compared to that of Student's t-test for testing the hypothesis that two normal populations with equal variances have equal means. To improve the power of Wilcoxon's test, it is suggested that a function of the ranks of the observations be used instead of the ranks themselves. In particular, a comparison of critical regions suggests the use of the statistic $X=\sum_1^g \psi(r/n+1)$, where ψ is the inverse of the standard normal cumulative distribution function and r stands for the ranks of the g observations from the first population in the overall ranking of the $g+h=n$ observations from both populations. (There is a misprint in the definition of X.) It is stated that in a forthcoming paper it will be shown that the X-test is asymptotically as powerful as Student's test.

Reviewer's note: A similar test has been suggested by Terry [Ann. Math. Statistics **23**, 346–366 (1952); these Rev. **14**, 190], who uses the statistic $c_1(R)=\sum_1^g E(Z_{nr})$, where Z_{nr} is the rth order statistic in a random sample of size n from a standard normal population. *G. E. Noether.*

Birnbaum, Z. W. Distribution-free tests of fit for continuous distribution functions. Ann. Math. Statistics **24**, 1–8 (1953).

This is a review article. The author brings out some of the essential features of various distribution-free tests of fit. Criteria for choosing an appropriate test are discussed briefly. *B. Epstein* (Detroit, Mich.).

Okamoto, Masashi. Unbiasedness in the test of goodness of fit. Osaka Math. J. **4**, 211–214 (1952).

Consider tests of the simple hypothesis H_0 that the cdf $F(x)=F_0(x)$ based on the number of observations $k_1, \cdots, k_n$ falling into n mutually exclusive and exhaustive intervals having equal probability according to H_0. Any such test corresponds to an acceptance region R in the space of n-dimensional lattice points $(k_1, \cdots, k_n)$. The set R is said to be symmetric if it is invariant under all permutations of the coordinates. It is said to satisfy condition 0 if for any point $(k_1, \cdots, k_i, \cdots, k_j, \cdots, k_n)$ with $k_j \geqq k_i+2$ it also contains the point $(k_1, \cdots, k_i+1, \cdots, k_j-1, \cdots, k_n)$.

The author proves the theorem: If the acceptance region R of the test is symmetric and satisfies condition 0, then the test is unbiased against any alternative. Application of the theorem to the Pearson chi-square test (with equal probabilities) and David's test [Biometrika **37**, 97–110 (1950); these Rev. **12**, 38] shows that these two tests are unbiased. *G. E. Noether* (Boston, Mass.).

Okamoto, Masashi. Some combinatorial tests of goodness of fit. Osaka Math. J. **4**, 215–228 (1952).

Given N observations $x_1, \cdots, x_N$ from a population with cdf $F(x)$ defined on the interval $0 \leqq x \leqq 1$. Denote by M_k the number of k-tuples $(x_{\alpha_1}, \cdots, x_{\alpha_k})$, $1 \leqq \alpha_1 < \cdots < \alpha_k \leqq N$, such that $x_{\alpha_1}, \cdots, x_{\alpha_k}$ fall in the same interval $((i-1)/n, i/n]$, $i=1, \cdots, n$. The author considers the test of the hypothesis H_0 that $F(x)$ is the uniform distribution which consists in rejecting H_0 if M_k is too large.

Let n, $N \to \infty$ such that $N/n \to$ constant (possibly ∞). It is shown that M_k is asymptotically normally distributed under H_0 as well as under the alternative hypothesis provided $F(x)$ has a density function $f(x)$ which is continuous except possibly at a finite number of points and such that $\int_0^1 f^m(x)dx$ $(m=2, 3, \cdots, 2k-1)$ exists. The test is shown to be consistent with respect to the same alternatives. It is unbiased against all alternatives.

The connections between the M_k-test and David's test [Biometrika **37**, 97–110 (1950); these Rev. **12**, 38] as well as the chi-square test (with equal probabilities) are established, proving, among other things, the consistency of the chi-square test with respect to the same alternatives as the M_2-test. *G. E. Noether* (Boston, Mass.).

Whittle, P. Tests of fit in time series. Biometrika **39**, 309–318 (1952).

Given two hypotheses, H_0 and H_1, of which H_0 is an instance of H_1 by fixing one or more parameters; denote the residual sums of squares under these hypotheses by U_0 and

U_1 respectively. Then $(U_0-U_1)/U_1$ is asymptotically distributed as χ^2, and serves as a statistic to compare the hypotheses. If two hypotheses are under study neither of which is an instance of the other, the author compares each with an accurate graduation (however redundant) obtained by a high order autocorrelation or similar method. The artificial example to which he applies this procedure is impressive, though it shows that the test depends on rather high accuracy in the graduation. Tests of this type yield approximations to the probability that the true deviation from H_0 is of the same sign as the observed deviation.

A. Blake (Buffalo, N. Y.).

Johnson, N. L. Comparison of analysis of variance power functions in the parametric and random models. Biometrika **39**, 427–429 (1952).

The power functions for models I and II for the analysis of variance are compared. Model I: $x_{ti}=A+B_t+z_{ti}$. Model II: $x_{ti}=A+u_t+z_{ti}$, $t=1,\cdots,k$; $i=1,\cdots,n$; A and the B_t are parameters, $\sum B_t=0$; the u's and z's are mutually independent normal random variables with 0 means; $Eu_t^2=\sigma'^2$; $Ez_{ti}^2=\sigma^2$. Let $S^2=\sum B_t^2/k$ for model I; models I and II are considered comparable if $S^2=\sigma'^2$. Let $\lambda_1=knS^2/\sigma^2$ and $\lambda_2=kn\sigma'^2/\sigma^2$ and let $\beta_I(\lambda_1)$ and $\beta_{II}(\lambda_2)$ be the respective power functions for the standard F-tests of the respective standard null hypotheses. Approximate work indicates that $\beta_I(\lambda)>\beta_{II}(\lambda)$ for $\lambda_1=\lambda_2=\lambda$ provided λ is sufficiently large, and that for the usual range of significance levels $(0.001\leqq\alpha\leqq0.10)$ β_I is slightly larger than β_{II} for small λ and approaches unity much more rapidly than β_{II} as λ increases. A conjecture of Patnaik [Biometrika **36**, 202–232 (1949); these Rev. **11**, 608] is thus mainly verified.

T. E. Harris (Santa Monica, Calif.).

Masuyama, Motosaburo. Graphical method of statistical inference. III. Approximate distribution of the ratio of two normalized variates. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. **1**, no. 4, 1–6 (1952).

The author continues the discussion of his improved binomial probability paper [same Rep. **1**, no. 2, 15–22 (1951); these Rev. **13**, 961]. The paper can be used to obtain approximately the power function for the F-test. The approximation is based on the fact that $(2x^2)^{1/2}$ is approximately normal with mean $(2f-1)^{1/2}$ and unit variance when the number of degrees of freedom f is large, and the fact that the ratio of two independent normal random variables is approximately normal if the mean of the denominator is sufficiently large.

T. E. Harris.

Preston, Glenn W. On the theory of prediction of non-stationary stochastic processes. J. Appl. Phys. **24**, 230–231 (1953).

Rios, Sixto. New applications of statistics: operational research. Trabajos Estadística **3**, 255–272 (1952). (Spanish)

Mathematical Economics

McKinsey, J. C. C. Some notions and problems of game theory. Bull. Amer. Math. Soc. **58**, 591–611 (1952).

This address, presented to the American Mathematical Society on December 1, 1951, provides a survey of the progress in the mathematical theory of games of strategy to that date. Explanatory in attitude, it includes an excellent description of the transition from games in extensive form to games in normalized form. With this background, the author gives a self-contained account of some of the important results for finite zero-sum two-person games and for infinite games played over the closed unit square. In addition, several areas with unsolved problems are discussed. These include infinite games played over function spaces, extensive "pseudo-games" in which an information axiom is violated, and the important subject of games which are not zero-sums, and which may have more than two players. Although no proofs are given, appropriate references are given whenever the material is available in the literature. Two changes may be made in the bibliography: (1) Item 9 should read: I. Glicksberg and O. Gross, "Notes on games over the square", Ann. of Math. Study No. 28, Princeton, 1953; (2) the results of L. S. Shapley mentioned in footnote 13 are contained in "A value for n-person games", ibid.

H. W. Kuhn (Bryn Mawr, Pa.).

***Raiffa, Howard. Arbitration schemes for generalized two-person games.** Contributions to the theory of games, vol. 2, pp. 361–387. Annals of Mathematics Studies, no. 28. Princeton University Press, Princeton, N. J., 1953. $4.00.

Motivated by the role of arbitration in collective bargaining disputes, we define a scheme of arbitration as a mapping from a given generalized two-person game into a unique element of the solution set of that particular game. We shall initially require that the domain of the arbitration mapping shall at least include all two-person games with a finite number of pure strategies. Various conditions are imposed on the arbitration schemes, and this paper shall attempt to construct mappings satisfying these desiderata. No attempt will be made on the formal level to single out a "best" method of arbitration; however, certain provocative games will be arbitrated according to different conventions to aid in the subjective evaluations of these conventions. The results and motivations for this paper were obtained independently of the work of J. F. Nash, Jr., on the two-person non-zero-sum game. (From the author's introduction.)

J. Wolfowitz (Urbana, Ill.).

Fréchet, Maurice. Emile Borel, initiator of the theory of psychological games and its application. Econometrica **21**, 95–96 (1953).

Borel, Emile. The theory of play and integral equations with skew symmetric kernels. Econometrica **21**, 97–100 (1953).

Translated from C. R. Acad. Sci. Paris **173**, 1304–1308 (1921).

Borel, Emile. On games that involve chance and the skill of the players. Econometrica **21**, 101–115 (1953).

Translated from Eléments de la théorie des probabilités, 3e éd., Hermann, Paris, 1924, pp. 204–221.

Borel, Emile. On systems of linear forms of skew symmetric determinant and the general theory of play. Econometrica **21**, 116–117 (1953).

Translated from C. R. Acad. Sci. Paris **184**, 52–54 (1927).

Fréchet, Maurice. Commentary on the three notes of Emile Borel. Econometrica **21**, 118–124 (1953).

In connection with the three notes listed above the author discusses the evolution of the theory of games, stressing the importance of Borel's contributions.

von Neumann, J. Communication on the Borel notes. Econometrica **21**, 124–127 (1953).

The author makes some further comments on the evolution of the theory of games and on Fréchet's discussion in the note listed above.

Hildreth, Clifford. Alternative conditions for social orderings. Econometrica **21**, 81–94 (1953).

The problem of social choice is to derive a social ordering over some space of social states from the orderings of a number of individuals. The individual orderings are each supposed to satisfy the von Neumann-Morgenstern axioms, random combinations of social states being also regarded as included in the basic space; it is further assumed that there exist two states $\bar{X}$ and $\bar{Y}$ such that each individual prefers $\bar{X}$ to $\bar{Y}$. In any given situation there is a set of social states achievable, given technical and resource limitations. It is assumed that such a set contains any random combination of any two social states in it; further, the sets have a kind of closure property relative to the individual orderings. It is desired to construct an ordering which is a function of the individual orderings and which satisfies the following properties: (1) for any set of achievable states having the above properties, a maximum element relative to the social ordering exists; (2) if each individual prefers X to Y or is indifferent between them and at least one is not indifferent, then the social ordering favors X to Y; (3) the social ordering is independent of the indexing of individuals; (4) if there exists a one-one mapping of the social states into themselves which maps the individual ordering of individual i into that of individual j and vice versa, while all other individuals are indifferent between a social state and its image, then, for any maximum element of a set of achievable social states relative to the social ordering, its image is also a maximum element if it is also in the same set. It is shown that the following class of social orderings satisfies all the above properties: let $f_i(X)$ be a von Neumann-Morgenstern utility index for individual i, so normalized that $f_i(X)=a$, $f_i(Y)=b$ for all i (where $a>b$), and let $g(u)$ be continuous, monotonic increasing, and strictly concave over the real numbers; then $\sum_{i=1}^{n} g[f_i(X)]$ is a utility function defining a social ordering satisfying all the above properties. It is not known whether or not other social orderings with these properties exist. *K. J. Arrow* (Stanford, Calif.).

***Koopmans, Tjalling C., and Reiter, Stanley. A model of transportation.** Activity Analysis of Production and Allocation, pp. 222–259. Cowles Commission Monograph No. 13. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. $4.50.

Let there be n ports and a transportation program which specifies the desired or required cargo movements between each pair of ports per unit time. The cost of moving a unit of cargo between any pair of ports is known, as is the cost of sailing empty. The problem is to find a routing plan which minimizes the total cost of the given program. If costs are measured in ship-time, total shipping in use is minimized. Since the given program specifies all those activities in which cargo is carried, the problem of optimization reduces to the choice of a routing plan for empty ships, to balance loaded movements. This "transportation problem" was one of the first linear programming problems to be explicitly treated [F. L. Hitchcock, J. Math. Physics **20**, 224–230 (1941); these Rev. **3**, 11]. Koopmans solved it independently, and the authors here handle it as an application of the general linear activity analysis model described by Koopmans elsewhere in the same volume [pp. 33–97; these Rev. **13**, 670].

As usual the efficient set in the commodity (cargo-movement) space is characterized by the existence of a vector of positive efficiency prices such that each activity engaged in has a zero profit and each available but unused activity has negative profitability. From these intrinsic prices on the final commodities are derived a set of efficiency prices on empty-ship appearances at different ports. If p_i and p_j are these prices at ports i and j and s_{ij} is the cost of sailing empty from i to j, then it must be that either $p_j=p_i+s_{ij}$ or $p_i=p_j+s_{ji}$ or $p_j<p_i+s_{ij}$ and $p_i<p_j+s_{ji}$, according as the optimal program involves a net flow of empty ships from i to j, or vice versa, or no flow between them at all. This is reminiscent of the classical Cournot problem of price equilibrium in spatially separated markets [P. A. Samuelson, American Economic Review **42**, 283–303 (1952)]. The efficiency prices p_{ij} on the cargo-carrying activity from i to j will satisfy: $p_{ij}-p_i+p_j-t_{ij}\leqq 0$, with the equality sign holding if and only if there is a positive flow of cargo from i to j. Here t_{ij} is the cost of moving a unit of cargo over the route (i, j).

The problem lends itself to visualization as a linear graph with ports as vertices and routes on which there is a positive flow of empty ships as arcs. If the graph is connected, the efficiency prices on ship appearances are all determined by any one of them. This determination can be carried out without contradiction according to the second paragraph above if and only if the graph describes an efficient solution to the problem. The disconnected case requires only slight modification.

Expressions like $t_{ij}+p_i-p_j$ and p_{ij} respectively are shown to give the marginal cost of an increase in traffic flow from i to j and the marginal rate of substitution of cargo flow against shipping. The efficiency prices are also interpreted as competitive market prices. The graph representation points up the analogy with electrical circuit theory. In fact the prices p_i can be regarded as the locational potential of an empty ship and in an efficient graph the potential drop between any two ports between which empty ships move is equal to the cost of a trip empty. The added difficulty of the transportation problem is that while Kirchhoff's laws follow from the minimization of a quadratic function, here the minimand is a linear form. *R. Solow.*

***Simon, Herbert A. Effects of technological change in a linear model.** Activity Analysis of Production and Allocation, pp. 260–277; comments, pp. 277–281. Cowles Commission Monograph No. 13. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. $4.50.

The author considers a special case of the activity analysis linear model of production in which there is only one final product. This avoids the index-number problem of measuring changes in real income. Within this framework he studies the effects of technological changes, i.e., changes in the production coefficients, and in the relative scarcities of the primary factors, on the choice of economical activities, and on the level of output of the single final commodity. Two cases must be distinguished: either the set of economical activities remains the same, after the change as before, or one or more previously used activities are replaced by ones not previously used. Some especially simple kinds of changes, such as a proportional reduction of all input coefficients in a particular activity, are analyzed. *R. Solow.*

TOPOLOGY

*Hönig, Chaim Samuel. **Sôbre um método de refinamento de topologias.** [On a method of refinement of topologies.] Thesis, University of São Paulo, São Paulo, 1952. viii+44 pp. (Portuguese)

A common method of constructing examples of curious topological spaces is to enlarge the family of open sets of a given topological space X by admitting as open all sets of the form $G \cap A'$, where G is open in X and A belongs to some specified ring $\mathfrak{A}$ of subsets of X. For example, where $X=[0,1]$ with its usual topology, one may take $\mathfrak{A}$ as all subsets of the set $\{\frac{1}{2}, \frac{1}{3}, \cdots, 1/n, \cdots\}$ or as all subsets of $[0,1]$ having Lebesgue measure 0. This basic idea has been systematically exploited in the present thesis. For a topology τ on a set E and any ring $\mathfrak{A}$ of subsets of E such that $A \in \mathfrak{A}$ implies that A has void interior under τ, let $\tau_{\mathfrak{A}}$ be the topology in which a basis for open sets consists of all sets $G \cap A'$, where G is open under τ and $A \in \mathfrak{A}$. Typical theorems are the following: If $\tau \leqq \tau' \leqq \tau_A$ (usual ordering), then $\tau' = \tau_{\mathfrak{B}}$ for some ring $\mathfrak{B}$ consisting of sets with void τ-interior. The collection of all topologies $\tau_{\mathfrak{A}}$ satisfies the hypothesis of Zorn's lemma, under the ordering $\leqq$. If F is a regular space, then a mapping f of E onto F is τ-continuous if and only if it is $\tau_{\mathfrak{A}}$-continuous, where $\mathfrak{A}$ is any ring of subsets as above.

The author reproduces much of the theory of semi-regular spaces expounded by M. H. Stone [Trans. Amer. Math. Soc. **41**, 375–481 (1937)] and the reviewer [Duke Math. J. **10**, 309–333 (1943); these Rev. **5**, 46]. His fundamental new result is that if τ_1 is a semi-regular topology on E and τ is any topology on E, then τ_1 is the semi-regular contraction of τ if and only if $\tau = \tau_{1\mathfrak{A}}$ for some ring $\mathfrak{A}$ of subsets of E having void τ_1-interiors. Next, applications of the foregoing theory are made. It is first shown that the semi-regular contraction of an absolutely closed [Hausdorff] space is a minimal Hausdorff space. If a given topology τ is absolutely closed, then every $\tau_{\mathfrak{A}}$ is also. The following interesting characterization of absolutely closed spaces is given: a topology τ is absolutely closed if and only if $\tau = (\tau^*)_{\mathfrak{A}}$ (τ^* is the semi-regular contraction of τ), where τ^* is a minimal Hausdorff space.

A theorem of the reviewer [loc. cit., Theorem 25] dealing with the existence of irresolvable spaces is re-proved; the methods of proof are very similar. Extremal disconnectivity (little new), extensibility of continuous mappings (interesting observations), and various special examples, conclude the thesis. *E. Hewitt* (Seattle, Wash.).

Shirota, Taira. A generalization of a theorem of I. Kaplansky. Osaka Math. J. **4**, 121–132 (1952).

Let X be a completely regular space, $C(X)$ the lattice of all real continuous functions on X. The author shows that a sublattice $K(X)$ of $C(X)$ characterizes X in the following cases. (1) X is locally compact, $K(X)$ = functions with compact support. (2) X is locally compact, $K(X)$ = functions which vanish at infinity. (3) X is locally compact and fully normal, $K(X)=C(X)$. (4) X is complete metric, $K(X)$ = uniformly continuous functions, or bounded uniformly continuous functions. (5) X is ϵ-complete (which means X is complete under the uniformity whose basis consists of all countable normal coverings), $K(X)=C(X)$. (6) $K(X)=C(X)$ topologized by any point-admissible topology weaker than the compact open topology. All these results generalize Kaplansky's theorem that $C(X)$ characterizes X if X is compact. *S. B. Myers.*

Fan, Ky, and Gottesman, Noel. On compactifications of Freudenthal and Wallman. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 504–510 (1952).

Let R be a regular space. A base $\mathfrak{B}$ for R is called a normal base if it has the properties: 1) $A, B \in \mathfrak{B}$ implies $A \cap B \in \mathfrak{B}$; 2) $A \in \mathfrak{B}$ implies $R - \bar{A} \in \mathfrak{B}$; 3) if U is an open set in R and $A \subset U$ with $A \in \mathfrak{B}$, then there is a $B \in \mathfrak{B}$ such that $A \subset B \subset \bar{B} \subset U$. The principal result is the theorem that if a regular space R has a normal base $\mathfrak{B}$, then R is homeomorphic to a dense subset of a compact Hausdorff space R^*, the $\mathfrak{B}$-compactification of R. In case R is normal, and $\mathfrak{B}$ is chosen as the set of all open subsets of R, one gets the Wallman compactification. The "points" of the space R^* are certain subsets of the elements of the base, called maximal binding families. If $A_1, \cdots, A_n$ is a finite collection, $A_i \in \mathfrak{B}$, $i=1, 2, \cdots, n$, belonging to a binding family, then the intersection of the closures of the A_i is non-vacuous. *L. Zippin* (Flushing, N. Y.).

Whyburn, G. T. A unified space for mappings. Trans. Amer. Math. Soc. **74**, 344–350 (1953).

Let $f: X \to Y$ be a continuous function on X onto Y and assume (for convenience of exposition) that $X \cap Y$ is void. The set $X \cup Y$ is topologized by saying that its subset Q is open if $Q \cap X$ and $Q \cap Y$ are open and if $K \subset Q \cap Y$ is compact, then $f^{-1}(K) \cap (X-Q)$ is compact. When X and Y are Hausdorff, then $X \cup Y$ is a T_1-space. Assume now that X and Y are Hausdorff. The function $r: X \cup Y \to Y$, defined by $r(z) = f(z)$ if $z \in X$ and $r(z) = z$ if $z \in Y$, is a retraction such that $r^{-1}(A)$ is compact when A is compact, and r is open if f is open. If X and Y are locally compact and perfectly separable, then so also is $X \cup Y$. Other results of a similar character are given. If (for example) both X and Y are compact, the condition $f^{-1}(K) \cap (X-Q)$ is a consequence of the conditions that $Q \cap X$ and $Q \cap Y$ be open. Thus X and Y are each open and closed in $X \cup Y$. *A. D. Wallace.*

Bing, R. H. A convex metric with unique segments. Proc. Amer. Math. Soc. **4**, 167–174 (1953).

It is shown that if M is a compact, metric, connected, locally connected space, then there is a dense subset W of M, and a convex distance function d, preserving the topology of M, such that each two points of W are the end-points of a unique segment. (Here a segment is a set isometric to a closed linear interval; d is convex if every two points of M are the end-points of a segment.) *E. E. Moise.*

Fuller, F. B. The existence of periodic points. Ann. of Math. (2) **57**, 229–230 (1953).

Two theorems are proved which for certain spaces X and certain maps $f: X \to X$ guarantee the existence of points periodic under f. Using properties of intersections of homology classes on manifolds, it is proved that if X is an orientable combinatorial manifold with non-zero characteristic and if f is a homeomorphism, then there is a periodic point. Using the Lefschetz fixed point theorem, the author proves a generalization as follows: if X is a finite complex with non-zero characteristic and if f induces isomorphisms of the rational homology groups, then f has a periodic point x. Moreover, x may be chosen so that its period is not greater than $\max(\sum_{i \text{ odd}} B_i, \sum_{i \text{ even}} B_i)$, where B_i is the ith Betti number of X. *E. E. Floyd* (Charlottesville, Va.).

Miščenko, E. F. On some questions of the combinatorial topology of nonclosed sets. Mat. Sbornik N.S. 32(74), 219–224 (1953). (Russian)

In the study of duality relations for non-closed subsets of the n-sphere S^n, Aleksandrov [Mat. Sbornik N.S. 21(63), 161–232 (1947); these Rev. 9, 456] and Kaplan [Trans. Amer. Math. Soc. 62, 248–271 (1947); these Rev. 9, 456] have introduced several homology groups for a subset E of S^n. In the notation of Aleksandrov, there is the group $\delta^p E$, its isomorph $D^p E$, as well as the group $\Delta_c{}^p E$ generated by projection cycles with compact carriers. The question has been raised by both Aleksandrov and Kaplan as to whether or not $D^p E$ and $\Delta_c{}^p E$ are always isomorphic. The author shows that they are not always isomorphic; he exhibits a simple example in S^3 with $D^1 E \neq 0$ and $\Delta_c{}^1 E = 0$. An example is also exhibited of a subset A of S^3 which, again in the notation of Aleksandrov, is a (ur^p, a^p)-set but whose complement is not a (J^q)-set. *E. E. Floyd.*

Eilenberg, Samuel, and MacLane, Saunders. Acyclic models. Amer. J. Math. 75, 189–199 (1953).

Formulation, dans le langage des catégories et des foncteurs, d'une situation qu'on rencontre en topologie algébrique et en algèbre. Considérons deux foncteurs covariants K et L, définis sur une catégorie $\mathfrak{A}$, à valeurs dans la catégorie des complexes; étant donnés, pour les entiers $n<q$, des homomorphismes $K_n \to L_n$ qui commutent avec les opérateurs différentiels de L et K, on cherche, pour tous les $n \geqq q$, des homomorphismes $K_n \to L_n$ de manière que la commutation avec les opérateurs différentiels soit assurée pour tous les degrés. Un théorème général (théorème II) donne des conditions suffisantes pour que ce problème ait une solution et que deux solutions quelconques soient "homotopes". On obtient ainsi aisément des démonstrations de résultats antérieurs des auteurs. En outre, ce théorème est appliqué à la comparaison des diverses théories d'homologie singulière, basées respectivement sur les simplexes et sur les cubes; c'est la première fois qu'une preuve complète de l'isomorphisme de toutes ces thèories est donnée dans la littérature.

H. Cartan (Paris).

Eilenberg, Samuel, and Zilber, J. A. On products of complexes. Amer. J. Math. 75, 200–204 (1953).

En utilisant le théorème II de l'article analysé ci-dessus, les auteurs prouvent le théorème suivant: si K et L sont deux complexes semi-simpliciaux complets, le "produit cartésien" $K \times L$ et le produit tensoriel $K \otimes L$ sont deux complexes homotopiquement équivalents. La démonstration montre l'existence d'applications naturelles (au sens des foncteurs) $f: K \times L \to K \otimes L$, $g: K \otimes L \to K \times L$, compatibles avec les opérateurs différentiels de ces complexes, et telles que fg et gf soient les applications identiques.

Première application: si X et Y sont deux espaces topologiques, le complexe singulier $S(X \times Y) = S(X) \times S(Y)$ est équivalent à $S(X) \otimes S(Y)$. Deuxième application: si K et L sont deux complexes simpliciaux, le "produit simplicial" $K \Delta L$ est homotopiquement équivalent à $K \otimes L$; pour le voir on associe à chaque complexe simplicial M un complexe semi-simplicial complet $O(M)$ (subdivision barycentrique) qui est équivalent à M, puis on observe que $O(K \Delta L) = O(K) \times O(L)$. Ceci peut s'appliquer notamment au nerf du "produit" de deux recouvrements de deux espaces.

H. Cartan (Paris).

***Séminaire de topologie algébrique de l'Ecole Normale Supérieure, 1948–49. Dix-sept exposés, avec pagination individuelle, par H. Cartan, J.-P. Serre, J. Cerf, P. Samuel, J. Dixmier, et J. Frenkel.** 167 pp. $2.00. (These unbound mimeographed notes may be ordered from: Mathematics Dept., Cornell University, Ithaca, N. Y.)

Contents: I. Complexes simpliciaux; II. Groupe d'homologie d'un complexe simplicial; III. Groupes abéliens discrets; IV. Dualité; V. Homologie et cohomologie singulières; VI. Cohomologie de Čech-Alexander; VII–IX. Opérateurs d'homotopie; X. Coefficients locaux; XI. Produits tensoriels; XII. Faisceaux et carapaces: définitions et exemples; XIII. Faisceaux (suite); XIV. Faisceaux fins; XV. Produits tensoriels de faisceaux; XVI. Théorèmes fondamentaux de la théorie des faisceaux; XVII. Homologie et cohomologie des variétés.

The first nine exposés constitute a more or less standard treatment of simplicial homology theory, singular homology theory, and the Čech-Alexander cohomology theory of locally compact spaces. Exposé X is an introduction to the Steenrod theory of local coefficients, and exposé XI contains a proof of the fundamental theorem on the homology groups of product spaces.

Exposés XII–XVII are devoted to an exposition of the Leray theory of "faisceaux" and "carapaces" on locally compact, paracompact spaces; this theory generalizes the notions of groups of chains and cochains with local coefficients [see Leray, J. Math. Pures Appl. (9) 29, 1–80, 81–139 (1950); these Rev. 12, 272]. Numerous applications of the theory of faisceaux are given; in particular, it is used in exposé XVII to study the homology theory (including duality, product and intersection theory) of a topological variety, i.e., a paracompact space locally homeomorphic with Euclidean space of a fixed dimension.

The material covered by exposés XII–XVII forms the subject matter of the same seminar, 1950–51, exposés XIV–XX [see the following review]; modifications of a substantial nature have, however, been introduced in the later treatment. *P. J. Hilton* (Cambridge, England).

***Séminaire de topologie algébrique de l'Ecole Normale Supérieure, 1950–51. Exposés XIV–XX par H. Cartan et exposé XXI par J.-P. Serre. Avec un appendice de deux exposés de H. Cartan du Séminaire de topologie algébrique de l'Ecole Normale Supérieure, 1949–50.** 98+i +29 pp. $1.50 (These unbound mimeographed notes may be ordered from: Mathematics Dept., Massachusetts Institute of Technology, Cambridge 39, Mass.)

Contents: XIV–XV. Faisceaux sur un espace topologique; XVI. Théorie axiomatique de la cohomologie; XVII. Théorie de la cohomologie des espaces; XVIII. Carapaces; XIX. Théorèmes fondamentaux de la théorie des faisceaux; XX. Théorie des faisceaux (application des théorèmes fondamentaux, étude de la structure multiplicative); XXI. La suite spectrale attachée à une application continue; Appendix: Cohomologie réelle d'un espace fibré principal différentiable.

Exposés XIV–XX are concerned with the same theory as that covered by exposés XII–XVII of the seminar of 1948–49 [see the preceding review; we refer to exposé N of this earlier seminar as N(48)]. However, the definitions of "faisceaux" and "carapaces" have been modified in such a way that greater generality is achieved and the role of Leray-Koszul spectral sequences [see the paper by Leray

cited in the preceding review] in the development of the theory is made explicit.

Exposé XIV gives the new definition and basic properties of a faisceau. If K is a principal ideal integral domain (e.g., a field or the ring of integers) and if X is a regular topological space, then a faisceau of K-modules over X is a topological space F, with which is associated a projection $p: F \to X$, which is a local homeomorphism such that, for all $x \in X$, $F_x = p^{-1}(x)$ is a topological K-module. The definition is strongly reminiscent of that of a fibre-space over X, and the analogy is enriched by the fundamental role played by the concept of a section of a faisceau over an open subset of X (i.e., a local right inverse of p), and that of the faisceau over Y induced by a map $f: Y \to X$. Various operations on faisceaux are defined, in particular, homomorphisms of faisceaux over the same and different spaces. The tensor product of a faisceau F over X and a faisceau G over Y is a faisceau $F \otimes G$ over $X \times Y$. If $X = Y$, then the diagonal map $X \to X \times X$ induces a faisceau $F \circ G$ over X, also called the tensor product of F and G (the module over x is the tensor product of the modules F_x and G_x). If U is an open subset of X, the set of sections over U, $\Gamma(F, U)$, is given the structure of a K-module in the obvious way.

Exposé XV defines and discusses (i) fine faisceaux, (ii) fundamental families, Φ, of closed sets of X, (iii) graduated differential faisceaux ("faisceaux gradués avec cobord"). A fine faisceau is one, roughly speaking, in which every element may be expressed as the sum of elements with small supports, "small" being measured with respect to some arbitrary locally-finite covering of X. A fundamental family Φ is a family of closed paracompact subsets of X such that the union of any two members of Φ is a member of Φ, and any closed subset of a member of Φ is a member of Φ. Defining the support of a section $s: U \to F$ as the set of all elements $x \in U$ such that $s(x) \neq 0 \in F_x$, the set of sections over X whose supports have (closed) neighbourhoods belonging to Φ, $\Gamma_\Phi(F)$, is a submodule of $\Gamma(F, X)$. (It is shown in a later exposé that there is no essential loss of generality in supposing that all closed neighbourhoods of members of Φ belong to Φ; we shall suppose this, so that $\Gamma_\Phi(F)$ becomes the submodule of sections whose supports belong to Φ.) Associated with an exact sequence of faisceaux and homomorphisms $0 \to F' \to F \to F'' \to 0$ is an exact sequence $0 \to \Gamma_\Phi(F') \to \Gamma_\Phi(F) \to \Gamma_\Phi(F'')$. The last homomorphism is not, in general, onto, but the main theorem of this exposé asserts that a homomorphism of F onto F'' induces a homomorphism of $\Gamma_\Phi(F \circ G)$ onto $\Gamma_\Phi(F'' \circ G)$ if G is fine. Given a graduated differential faisceau (g.d.f.) F, defined in the natural way, we may define the subfaisceaux $Z(F)$, $B(F)$ of cocycles and coboundaries. The quotient faisceau $H(F)$ is called the cohomology faisceau of F, and is graduated. The structure on F gives rise to a graduated differential structure on $\Gamma_\Phi(F)$, whence we may construct the cohomology module $H(\Gamma_\Phi(F))$. Three examples of graduated differential faisceaux are given; the proofs of the triviality of the resulting cohomology faisceau are given in VII(48) and XII(48).

Exposé XVI presents an axiomatic cohomology theory on a regular space. The theory associates with a given space X, integer q, family Φ, and "coefficient" faisceau F (over X), a K-module $H_\Phi^q(X, F)$; with a faisceau-homomorphism $F \to F'$ is associated a module-homomorphism

$$H_\Phi^q(X, F) \to H_\Phi^q(X, F');$$

and with an exact sequence $0 \to F' \to F \to F'' \to 0$ is associated a module-homomorphism $H_\Phi^q(X, F'') \to H_\Phi^{q+1}(X, F')$. This system of modules and homomorphisms is subjected to six axioms, of which the only one unfamiliar in classical cohomology theory is that which asserts that $H_\Phi^q(X, F) = 0$ if $q \neq 0$, F being fine. In fact, the resulting theory reduces to the Čech theory if X is compact, Φ the family of all closed subsets of X, and F a "simple" faisceau. It is, therefore, called the Čech cohomology theory (relative to Φ and F). In the theory presented in XII–XVII(48), Φ was restricted to the family of all closed subsets of X or the family of all compact subsets of X. The existence and uniqueness of a Čech cohomology theory is proved in this exposé. The existence part of the proof yields a method of constructing a cohomology theory by identifying $H_\Phi^q(X, F)$ with $H^q(\Gamma_\Phi(C \circ F))$, where C is a graduated differential faisceau of "Alexander-Spanier cochains", whose cohomology is trivial. A suitable faisceau C is said to be Φ-fundamental (a precise definition is somewhat involved).

Exposé XVII is concerned with the cohomology sequence, with the notion of cohomology dimension, and with multiplication in cohomology. A faisceau F over X induces a faisceau F' over a subspace X'. Conversely, given a faisceau F' over a closed (or open) subspace X' of X, there exists a unique faisceau $\bar{F}'$ over X, such that $\bar{F}'$ induces F' over X' and the zero faisceau over $X'' = X - X'$. Let X' be open; then $\bar{F}'$ is a subfaisceau of F and $F/\bar{F}'$ may be identified with $\bar{F}''$, where F'' is the faisceau over X'' induced by F. By a cohomology axiom, the exact sequence

$$0 \to \bar{F}' \to F \to \bar{F}'' \to 0$$

gives rise to the exact sequence

$$\cdots \to H_\Phi^q(X, \bar{F}') \to H_\Phi^q(X, F) \to H_\Phi^q(X, \bar{F}'') \to H_\Phi^{q+1}(X, \bar{F}') \to \cdots,$$

which may be replaced, using natural isomorphisms, by the exact sequence

$$\cdots \to H_{\Phi'}^q(X', F') \to H_\Phi^q(X, F) \to H_{\Phi''}^q(X'', F'') \to H_{\Phi'}^{q+1}(X', F') \to \cdots,$$

where Φ', Φ'' are the subfamilies of Φ whose member-sets belong to X', X'', respectively. A graduated faisceau C is said to have dimension $\leqq n$ if C_q (its q-component) $= 0$, $q > n$. The main theorem of this exposé establishes necessary and sufficient conditions that a space X admit Φ-fundamental faisceaux of dimension $\leqq n$; of particular interest is the fact that a sufficient condition is the vanishing of $H_\Phi^{n+1}(X, F)$ for all coefficient faisceaux F. Let C_i be Φ_i-fundamental, $i = 1, 2$, and let Φ be the intersection of the families Φ_1, Φ_2. Then $C_1 \circ C_2$ is Φ-fundamental, and the canonical homomorphism $\Gamma_{\Phi_1}(C_1 \circ F_1) \otimes \Gamma_{\Phi_2}(C_2 \circ F_2) \to \Gamma_\Phi(C_1 \circ C_2 \circ F_1 \circ F_2)$ gives rise to a homomorphism

$$H_{\Phi_1}^p(X, F_1) \otimes H_{\Phi_2}^q(X, F_2) \to H_\Phi^{p+q}(X, F_1 \circ F_2).$$

In particular, if F is a faisceau of algebras, so that a homomorphism $F \circ F \to F$ is defined, $H_\Phi(X, F)$ takes on the structure of a graduated algebra, the cohomology algebra of X (with respect to Φ, F). The faisceau of Alexander-Spanier cochains is itself a faisceau of algebras (so, also, is the faisceau of differential forms on a differentiable variety, multiplication being the exterior multiplication of differential forms). In such a case we get a homomorphism $\Gamma_{\Phi_1}(C \circ F_1) \otimes \Gamma_{\Phi_2}(C \circ F_2) \to \Gamma_\Phi(C \circ F_1 \circ F_2)$, defined by the multiplicative structure of C. If, also, F is a faisceau of algebras, $\Gamma_\Phi(C \circ F)$ becomes a differential algebra, whose cohomology algebra coincides with the cohomology algebra of X.

In exposé XVIII the notion of "carapace" is defined. Given the space X and the ring K, a précarapace A is a K-module such that with each $x \in X$ is associated a homomorphism, ϕ_x, of A onto a quotient-module A_x such that,

if $\alpha \in A$ and $\phi_x\alpha=0$, then $\phi_y\alpha=0$ for all y sufficiently close to x. If, moreover, $\phi_x\alpha=0$, all $x \in X$, implies $\alpha=0$, then A is a carapace [cf. the definition of a "reduced faisceau" in Leray, loc. cit.]. If we define the support of α as the set of points $x \in X$ such that $\phi_x(\alpha) \neq 0$, then the supports are closed sets, and we may characterize a précarapace in terms of certain properties of the supports [Leray, loc. cit.]; in particular, a carapace is obtained from a précarapace by factoring by the elements with empty supports. It is not the case, as it was with the definitions used in the seminar of 1948–49, that a carapace is a special faisceau; however, with every carapace A is associated the faisceau $F(A)=\bigcup A_x$ (suitably topologized) and, conversely, every faisceau F gives rise to the carapace of sections $\Gamma(F)=\Gamma(F, X)$. The notion of graduated differential carapace (g.d.c.) is defined and $\Gamma(F)$ is a graduated differential carapace if F is a graduated differential faisceau, $F(A)$ is a graduated differential faisceau if A is a graduated differential carapace. The carapace A may be embedded isomorphically in $\Gamma(F(A))$; this is called completing A, and A is complete if it is its own completion. If A_Φ is the subcarapace of A consisting of elements whose supports belong to Φ, then the embedding of A in $\Gamma(F(A))$ embeds A_Φ in $\Gamma_\Phi(F(A))$. This leads to the notion of Φ-completion. The remainder of the exposé is devoted to extending the fundamental notions from faisceaux to carapaces (thus, tensor product of carapaces, induced carapaces, fine carapaces—this last concept is generalized in two important directions, to Φ-fine carapaces and homotopically fine carapaces).

The fundamental theorems of exposé XIX relate the "global" cohomology of the carapace of sections of a graduated differential faisceau F to the "local" properties of the faisceau of cohomology $H(F)$. Thus Theorem 3 asserts that if the cohomology of $H_\Phi^p(X, F)$ (with respect to the coboundary operator in F) is trivial for $p \geqq 1$, and if either X is of finite (cohomology) dimension or the degrees of F are bounded below, then there exists a spectral sequence $E_0, E_1, E_2, \cdots, E_\infty$, such that $E_2=\sum_{p,q} H_\Phi^p(X, H^q(F))$ and E_∞ is the graduated module associated with the module $H(\Gamma_\Phi(F))$, suitably filtered. This theorem is readily extended to Φ-complete carapaces A, $H(\Gamma_\Phi(F))$ being replaced by $H(A_\Phi)$. Theorem 4, which is a ready consequence of Theorem 3, asserts that if F and F' are graduated differential faisceaux over X, satisfying the conditions of Theorem 3 and if the homomorphism $H(F) \to H(F')$ induced by a homomorphism $f: F \to F'$ is an isomorphism onto, then so is the homomorphism $H(\Gamma_\Phi(F)) \to H(\Gamma_\Phi(F'))$ induced by f. This theorem, due essentially to Leray, appeared in XVI(48) in a number of special cases. The methods of proof used in this exposé will be familiar to those who have studied the application of filtered groups to the elucidation of homology relations between fibre-space, base-space, and fibre.

In exposé XX, the theorems of exposé XIX are applied, especially Theorem 4. It is shown that (i) for HLC (homology-locally-connected)-spaces, there is an isomorphism of the Čech cohomology theory with the singular cohomology theory (based on the faisceau of singular cochains); (ii) if X is a differentiable variety, the faisceau of differentiable cochains gives rise to a "differentiable" cohomology theory, isomorphic to the singular (and Čech) theories; similarly, starting from the faisceau of differentiable chains, the resulting "differentiable" homology theory is isomorphic to the singular theory; and, finally, (iii) if, in addition, K is taken to be the real field, then the faisceau of exterior differential forms defines a cohomology theory isomorphic to the others discussed. Duality and intersection theorems in a topological variety are deduced [see XVII(48)], and the exposé closes with a discussion of the multiplicative structure in cohomology. It is shown that the isomorphisms given by the fundamental theorems of exposé XIX respect the multiplicative structures where defined; a cap-product (cohomology-homology) is also defined.

Exposé XXI is a readable account of Leray's application of the foregoing theory to the study of the homology relations associated with a map $f: X \to Y$. Let Ψ, Φ be families of closed subsets of X, Y, respectively, and let G be an arbitrary faisceau over X. Then the main theorem of this exposé asserts that, under certain conditions on Φ, Ψ, there exists a spectral sequence such that $E_2^{p,q}=H_\Phi^p(Y, H_\Psi^q(F_y, G))$ and E_∞ is the graduated module associated with $H_\Psi(X, G)$ suitably filtered. Here we mean by $H_\Psi(F_y, G)$ the faisceau over Y formed by the cohomologies of the "fibres" $f^{-1}(y)=F_y$, $y \in Y$, with respect to the families induced in F_y by Ψ and the coefficient faisceaux induced in F_y by G. The main tool in the proof is Theorem 3 of exposé XIX. Unfortunately, changes have been made in notation—in particular, $E_2^{p,q}$ is written in this exposé $E^2_{p+q,p}$.

The appendix is not an appendix at all; it is simply the last two exposés of the seminar of 1949–50, and its appearance at the end of this set of exposés is mysterious to the reviewer. It is to be hoped that the two exposés concerned will be reunited with exposés I–XVIII of the seminar of 1949–50 when the latter are made generally available. The seminar of 1949–50 was devoted to the topics of homotopy theory (especially the homotopy groups of Lie groups), Steenrod squares, and fibre-spaces, and the last two exposés would be more appropriately reviewed against the background of a general treatment of fibre-spaces. They contain a discussion of infinitesimal connexions and characteristic classes in a principal fibre-space, the Weil algebra of a Lie group (or, better, of a Lie algebra), transgressions in Lie groups and principal fibre-spaces, and the cohomology of the base-space (and hence of homogeneous spaces). Perhaps this last is held to establish a tenuous connection with the material reviewed above. *P. J. Hilton.*

Massey, W. S. Exact couples in algebraic topology. I, II. Ann. of Math. (2) **56**, 363–396 (1952).

Ce travail comprend 2 parties: (I) la théorie algébrique générale des "exact couples" due à l'auteur [cf. Bull. Amer. Math. Soc. **57**, 281 (1951), abstract 323t]; (II) l'étude d'un couple exact particulier, qui conduit à des relations entre l'homologie et l'homotopie d'un espace et introduit de nouveaux invariants topologiques. D'autres applications feront l'objet d'un mémoire ultérieur.

Dans la partie (I), l'auteur généralise la notion de "suite spectrale" (ou "suite de Leray-Koszul") et montre comment la donnée d'un "couple exact" conduit à une suite spectrale. Dans la partie (II), on attache à tout espace topologique (pourvu que ce soit un *CW*-complexe) un couple exact et par conséquent une suite spectrale. L'auteur croit peu probable que cette suite spectrale puisse être obtenue à partir d'un groupe différentiel filtré; toutefois, le rapporteur signale qu'elle est justiciable du processus général exposé par S. Eilenberg [cf. exposé VIII de Séminaire de topologie algébrique de l'Ecole Normale Supérieure, 1950-51]. Soient K un *CW*-complexe, K^p le p-squelette; à tout couple (K^p, K^q) tel que $q<p$, associons le groupe d'homotopie relatif $\pi_n(K^p, K^q)$, convenablement modifié pour les petites valeurs de n, p, q; ces groupes, les homomorphismes d'inclusion $\pi_n(K^p, K^q) \to \pi_n(K^r, K^s)$ $(p \leqq r, q \leqq s)$, et les homomorphismes $\pi_n(K^p, K^q) \to \pi_{n-1}(K^q, K^r)$ $(p>q>r)$

définissent une suite spectrale au sens d'Eilenberg. La plus grande partie du mémoire est consacrée au calcul (partiel) du premier terme de cette suite spectrale (le terme $E^2_{p,q}$ dans la notation de Serre [Ann. of Math. (2) **54**, 425–505 (1951); ces Rev. **13**, 574]); noté $\mathfrak{C}_{p,q}(K)$ par l'auteur.

Voici quelques-uns des principaux résultats: $\mathfrak{C}_{p,q}(K)$ est un invariant du type d'homotopie de l'espace K, et ne dépend que du revêtement simplement connexe $\tilde{K}$. On a $\mathfrak{C}_{p,0}(K)=H_p(\tilde{K})$ (groupe d'homologie). Supposons $\pi_i(K)=0$ pour $1\leqq i\leqq r$; alors $\mathfrak{C}_{p,q}(K)=0$ pour $p\leqq r$; pour $q<r$, $\mathfrak{C}_{p,q}(K)\approx H_p(K, \mathfrak{G}_q)$, $\mathfrak{G}_q$ désignant le groupe stable $\pi_{n+q}(S^n)$ $(n\geqq q+2)$; pour $p>r+3$,

$$\mathfrak{C}_{p,q}(K)\approx H_p(K, \mathfrak{G}_q)+H_p(K, \pi_{q+1}(K)).$$

Enfin, pour tout CW-complexe K, $\mathfrak{C}_{2,1}(K)\approx\Gamma(\pi_2(K))$ (Γ: groupe de Whitehead) $\approx H_4(\pi_2(K), 2)$ (groupe d'Eilenberg-MacLane).

Le mémoire est suivi d'un appendice qui est un formulaire commode des groupes d'homotopie. *H. Cartan.*

Dolbeault, Pierre. Sur la cohomologie des variétés analytiques complexes. C. R. Acad. Sci. Paris **236**, 175–177 (1953).

Let M be a complex-analytic manifold, and let d be the operator of exterior differentiation. Then $d=\partial+\bar{\partial}$, where ∂ is the differential which maps forms of type (r, s) into forms of type $(r+1, s)$, $\bar{\partial}$ the differential which maps forms of type (r, s) into forms of type $(r, s+1)$. Denote by $A^{r,s}$ the faisceau of currents of type (r, s), by $F^{r,s}$ the faisceau of currents of type (r, s) which are closed under $\bar{\partial}$, by $\Omega^p(=F^{p,0})$ the faisceau of holomorphic p-forms, and by Φ^p the subfaisceau of Ω^p composed of holomorphic p-forms closed under d. Every faisceau $\mathfrak{F}$ of a vector space defines vector spaces $H^q(M, \mathfrak{F})$ of cohomology [cf. exposé III of Séminaire de topologie algébrique de l'Ecole Normale Supérieure, 1950–51].

Let $H^{p,q}(M)$ denote the group defined by elements of type (p, q) in the $\bar{\partial}$-cohomology of currents, B^p the graded faisceau which is equal to the direct sum $\sum_{r,s\geqq 0}A^{p+r,s}$, $\mathfrak{B}^p$ the vector space of sections of B^p, $H(\mathfrak{B}^p)$ the d-cohomology of $\mathfrak{B}^p$, $K^{p,q}(M)$ the subspace of $H(\mathfrak{B}^p)$ composed of elements of degree $p+q$.

For every pair of integers $p, q\geqq 0$ the author proves the existence of canonical isomorphisms:

$$(1)\qquad H^q(M, \Omega^p)\leftrightarrow H^{p,q}(M);$$

$$(2)\qquad H^q(M, \Phi^p)\leftrightarrow K^{p,q}(M).$$

The proofs are based on the complex analogue of the Poincaré lemma which asserts that the local $\bar{\partial}$-cohomology of currents of type (r, s), $s>0$, is trivial. This lemma, which is easily proved, implies that the sequence

$$0\to F^{r,s}\xrightarrow{i}A^{r,s}\xrightarrow{\bar{\partial}}F^{r,s+1}\to 0$$

(i=inclusion) is exact, and it follows that the cohomology sequence

$$\cdots\to H^{q-s-1}(M, A^{r,s})\to H^{q-s-1}(M, F^{r,s+1})\to H^{q-s}(M, F^{r,s})\to H^{q-s}(M, A^{r,s})\to\cdots$$

is exact for $q>s$. Since $A^{r,s}$ is a fine faisceau, $H^q(M, A^{r,s})=0$ for $q>0$ and the first result follows immediately. The same reasoning shows that $H^{q-s}(M, F^{r,s})$ and $H^{r,q}(M)$ are canonically isomorphic for $s>0$ provided that $q>s$. The second result is proved in a similar fashion. *D. C. Spencer.*

Miller, Clair E. The topology of rotation groups. Ann. of Math. (2) **57**, 90–114 (1953).

Notations: $R(n)$ désigne le groupe des rotations dans l'espace euclidien réel E^n [groupe $SO(n)$ dans la notation de Chevalley, Theory of Lie groups, v. I, Princeton, 1946; ces Rev. **7**, 412]; $V(n, r)$ est la variété de Stiefel, espace des r-repères orthonormaux de E^n; on a $R(n)\approx V(n, n-1)$.

L'auteur donne une décomposition cellulaire de $V(n, r)$ qui permet le calcul de la structure multiplicative de l'homologie de $R(n)$ et de la structure multiplicative de la cohomologie de $V(n, r)$. Cette décomposition est obtenue ainsi: un point de $P(n-1)$, espace projectif réel de dimension $n-1$, définit une droite de E^n passant par 0; le composé d'une symétrie par rapport à l'hyperplan orthogonal à cette droite, et d'une symétrie par rapport à un hyperplan fixe, est un élément de $R(n)$. D'où un homéomorphisme f de $P(n-1)$ dans $R(n)$. Pour toute suite d'entiers $b_1, \cdots, b_k$ $(1\leqq b_1<b_2<\cdots<b_k\leqq n)$, l'application

$$(x_1, \cdots, x_k)\to f(x_k)\cdots f(x_1)$$

est un homéomorphisme de $P(b_1)\times\cdots\times P(b_k)$ dans $R(n)$. En composant avec l'application fibrée p: $R(n)\to V(n, r)$, de fibre $R(n-r)$, on trouve un homéomorphisme de $P(b_1)\times\cdots\times P(b_k)$ sur une cellule de $V(n, r)$; les intérieurs de ces cellules constituent une décomposition cellulaire de $V(n, r)$. L'auteur explicite la formule du bord.

L'application $R(n)\times R(n)\to R(n)$ définie par la loi de groupe est cellulaire et permet d'expliciter le produit de Pontrjagin des chaînes; d'où la structure multiplicative de l'homologie. Par exemple, l'algèbre d'homologie de $R(n)$ à coefficients mod 2 est une algèbre extérieure engendrée par l'unité et des éléments de degrés $1, 2, \cdots, n-1$. On obtient aussi une formule de multiplication des cochaînes de $V(n, r)$. L'algèbre de cohomologie mod 2 est une algèbre abélienne engendrée par 1 et des éléments u_i uniquement assujettis à la condition $(u_i)^{k_i}=0$, k_i étant la plus petite puissance de 2 au moins égale au quotient de n par le degré de u_i; les degrés des u_i sont distincts, ce sont, outre $n-r$, les entiers impairs $>n-r$ et $<n$. L'algèbre de cohomologie rationnelle est aussi explicitée.

Enfin, l'auteur détermine les carrés de Steenrod de l'espace $V(n, r)$ et en déduit une démonstration d'un théorème connu de Steenrod et J. H. C. Whitehead [Proc. Nat. Acad. Sci. U. S. A. **37**, 58–63 (1951); ces Rev. **12**, 847]. *H. Cartan* (Paris).

Aoki, Kiyoshi. On maps of a $(2n-1)$-dimensional sphere into an n-dimensional sphere. Tôhoku Math. J. (2) **4**, 178–186 (1952).

Let n be even and f: $S^{2n+1}\to S^n$ have Hopf invariant zero. The author's theorem: If f restricted to the inverse image of a principal simplex of S^n satisfies some homotopy condition relative to the inverse image of an interior point, then f is nullhomotopic. The definitions of the homotopy condition and the auxiliary concepts involved are very ambiguous; the reviewer was unable to find interpretations which led to the statement of a theorem that was not patently false. The "proof" does not clarify any of these details. One thing seems certain: no use is made of the fact that f can be taken as the suspension of a g: $S^{2n-2}\to S^{n-1}$.

J. Dugundji (Princeton, N. J.).

Shimada, Nobuo. Homotopy classification of mappings of a 4-dimensional complex into a 2-dimensional sphere. Nagoya Math. J. **5**, 127–144 (1953).

The main purpose of this paper is to solve the homotopy classification problem mentioned in the title. To achieve

this, the author introduces several cohomology operations which are new topological invariants. First he defines an operation which associates with any r-dimensional cochain a and s-dimensional cochain b an $(r+s+1)$-dimensional cochain $a \vee b$. Using this cochain operation for the case $r=s=2$, he then defines an invariant cohomology operation, ψ, which associates with any 2-dimensional integral cohomology classes u and v such that $u \smile v = 0$ a coset, $\psi(u, v)$, of the 5-dimensional cohomology group (with coefficients modulo 2) modulo a certain subgroup. This new operation ψ is symmetric, but it is not bilinear. It is then shown that this new operation suffices to determine the third obstruction for the case of mappings into a 2-sphere, S^2. To be more precise, let K be a finite polyhedron, and let K^n denote its n-skeleton. Let $f: K^3 \to S^2$ be a continuous map which can be extended over K^4. Then the obstruction to extending f over K^5 is the coset $\psi(c^3(f), c^3(f))$, where $c^3(f) \in H^3(K)$ is the characteristic class of f. From this extension theorem, the author then derives a homotopy classification theorem for maps of a 4-complex into a 2-sphere by standard techniques.

In a note added in proof, the author acknowledges that this homotopy classification theorem has been found independently by J. Adem [Proc. Nat. Acad. Sci. U. S. A. **38**, 720–726 (1952); these Rev. **14**, 306]. Actually, this note of Adem contains an announcement (without proof) of his solution of the classification problem for maps of an $(n+2)$-complex into an n-sphere for $n>2$. However, a slight modification of Adem's methods also gives the classification theorem for the case $n=2$. *W. S. Massey.*

Uehara, Hiroshi. On homotopy type problems of special kinds of polyhedra. I. Osaka Math. J. **4**, 145–168 (1952).

According to the terminology of J. H. C. Whitehead [Ann. Soc. Polon. Math. **21**, 176–186 (1948); these Rev. **11**, 48] a finite polyhedron P is an A_n^p-polyhedron if the homotopy groups $\pi_i(P)$ are trivial for $i<n$, and the dimension of P is $\leqq n+p$. The main result of this paper asserts that for $n>3$, the homotopy type of an A_n^3-polyhedron P for which $\pi_{n+1}(P)=0$ is completely determined by its cohomology groups, its Steenrod squares, and certain standard homomorphisms between cohomology groups with different coefficients. The methods of proof are similar to those used by J. H. C. Whitehead in the paper cited above.

W. S. Massey (Providence, R. I.).

Uehara, Hiroshi. On homotopy type problems of special kinds of polyhedra. II. Osaka Math. J. **4**, 169–184 (1952).

In this paper, the author shows how the homotopy groups $\pi_{n+1}(P)$ and $\pi_{n+2}(P)$ are determined for an A_n^3-polyhedron P $(n>3)$ by means of certain homology groups and certain homomorphisms of these homology groups. The following theorem is also proved. Let P be a finite, simply connected polyhedron which has no two-dimensional torsion, and for which $\pi_3(P)=0$. Then the rank of $\pi_4(P)$ is $R_4-\frac{1}{2}R_2(R_2-1)$, and the torsion coefficients of $\pi_4(P)$ are the same as those of $H_4(P)$, the 4-dimensional integral homology group (here R_n denotes the nth Betti number). *W. S. Massey.*

Yoneda, Nobuo. Simplicial mapping, isotopy and desingularization concerning cylinders. Sûgaku **4**, 1–17 (1952). (Japanese)

This paper is concerned mainly with two concepts, "isotopy", and "desingularization". The author defines a concept of combinatorial isotopy of two complexes in a complex. The concept of desingularization may be explained as follows. If K is an m-dimensional simplicial complex, k is an n-dimensional simplicial complex, and $f: K \to k$ is a simplicial map, then there always exist simplexes whose dimension is reduced by at least $d=m-n$. If K' and k' denote subdivisions of K and k respectively, a simplicial map $f': K \to k'$ is called a desingularization of f in case f' is homotopic to f and no simplex is reduced in dimension more than $d=m-n$. In case K is a homogeneous m-dimensional complex, and k is an n-dimensional homotopy manifold, such a desingularization always exists.

These concepts provide methods for proving the homotopy invariance of some properties of a simplicial map. As some applications, the author proves classical theorems about the inverse homomorphism (Umkehrhomorphismus) and linking coefficients (Verschlingungszahl). No new results are given. *H. Uehara* (Nagoya).

GEOMETRY

Thébault, Victor. Un chapitre de géométrie du triangle et du tétraèdre. Orthopôle d'une droite. Bull. Soc. Roy. Sci. Liège **21**, 369–384 (1952).

An expository paper. *N. A. Court* (Norman, Okla.).

Thébault, Victor. On cevians of a triangle. Amer. Math. Monthly **60**, 167–173 (1953).

Thébault, V. Sur le quadrilatère convexe inscriptible. Math. Gaz. **37**, 14–17 (1953).

Goormaghtigh, R. Sur le quadrilatère complet. Mathesis **61**, supplement, 1–16 (1952).

Weitzenböck, R. W. Some remarks on equilateral triangles and squares. Math. Gaz. **37**, 21–26 (1953).

Inglada, Vicente. Pairs of related triangles studied by complex coordinates. Math. Gaz. **37**, 1–6 (1953).

Niče, Vilko. Contribution à la géométrie du tétraèdre. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **7**, 228–243 (1952). (Serbo-Croatian. French summary)

Roeser, Ernst. Corresponding polyhedra in the three spaces of constant curvature. Canadian J. Math. **5**, 40–45 (1953).

Formulae are obtained for the radii and angles of the general regular polyhedron in spherical, Euclidean, and hyperbolic 3-spaces, expressed in terms of the mid-radius, i.e., the distance from the center to an edge. Polyhedra of mid-radii ρ_s, ρ_e, ρ_h in the respective spaces are said to "correspond" if their numerical properties are the same while $\sin \rho_s = \rho_e = \sinh \rho_h$. This correspondence is exhibited geometrically in hyperbolic 4-space, where the spherical polyhedron is drawn on a hypersphere, the Euclidean polyhedron on a hyper-horosphere, and the hyperbolic polyhedron on a hyperplane, so situated that the edges have the same mid-points in all three cases. *H. S. M. Coxeter.*

Krames, Josef. Zur Geometrie der gegenseitigen Einpassung von Luftaufnahmen. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **160**, 113–128 (1951).

The author continues his earlier studies [Österreich. Z. Vermessgswes. **37**, 13–29 (1949); Schweiz. Z. Vermessg. Kulturtech. **49**, 7–16 (1951); these Rev. **12**, 135, 541] of the mutual orientation of aerial photographs. His aim is the improvement of the accuracy of the fit. *E. Lukacs.*

Baur, Arnold. Die Brennpunkte der Kegelschnitte. Math.-Phys. Semesterber. **3**, 80–86 (1953).

Room, T. G. The composition of rotations in Euclidean three-space. Amer. Math. Monthly **59**, 688–692 (1952).

The rotation about an axis through the origin parallel to the unit vector u through an angle 2δ is represented by the vector $\lambda = u \tan \delta$. For the resultant λ of two rotations λ_1 and λ_2 the author proves the theorem

$$\lambda = (\lambda_1 + \lambda_2 - \lambda_1 \times \lambda_2)/(1 - \lambda_1 \cdot \lambda_2).$$

Applications to spherical geometry. *O. Bottema* (Delft).

*Wolkowitsch, D. **Sur les applications de la notion de moment d'inertie en géométrie.** Mémor. Sci. Math., no. 121. Gauthier-Villars, Paris, 1952. 41 pp. 450 francs.

Etant donné un système rigide de masses on mène par son centre d'inertie un plan variable π, dont le rayon de giration est ρ; le plan parallèle à π, à la distance ρ, enveloppe l'ellipsoïde d'inertie E de Culmann. Si son équation tangentielle est $a^2u^2+b^2v^2+c^2w^2-s^2=0$, celle de la quadrique conjugée ϵ est $a^2u^2+b^2v^2+c^2w^2+s^2=0$. Les plans qui admettent un rayon de giration constant ρ, enveloppent les surfaces homofocales de ϵ (nommées isogires). Toute quadrique peut être considérée comme l'isogire d'un certain système. Ce théorème est le point fondamental des recherches de l'auteur, qui réussit à démontrer par ses considérations "inertiques" bon nombre de propriétés classiques et nouvelles de la géométrie supérieure. Nous ne donnons qu'une seule exemple. Cherchons le lieu des sommets des trièdres trirectangles dont les faces sont tangentes à trois quadriques d'une famille de quadriques homofocales. Chacune de ces trois quadriques est une enveloppe de plans de moment d'inertie constant. Soient ρ_1, ρ_2, ρ_3 les rayons de giration correspondants. Il est clair que pour un point du lieu A le moment d'inertie polaire est constant (le carré du rayon de giration polaire étant $\rho^2=\rho_1^2+\rho_2^2+\rho_3^2$). On conclut sans peine que le lieu est la sphère $(0, r)$, où $r^2=\rho^2-(a^2+b^2+c^2)$ et on a démontré avec simplicité une généralisation d'un théorème classique de Monge.

Citons parmi les autres résultats de l'auteur: beaucoup de théorèmes sur la sphère de Monge et une étude sur le complex de Painvin (qui peut être défini comme le lieu des droites pour lesquelles le rayon de giration est constant).

Au sens du ref. il faut lire au p. 5 partout "ligne de courbure" au lieu de "géodésique". *O. Bottema* (Delft).

Fletcher, T. J. Finite geometry by coordinate methods. Math. Gaz. **37**, 34–38 (1953).

Rajčić, Lav. Etude sur les constructions et les théorèmes fondamentaux de la géométrie d'espace de Lobatchevsky par les méthodes de la géométrie projective. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **7**, 66–87 (1952). (Serbo-Croatian. French summary)

This exposition of hyperbolic geometry in space is a continuation of an earlier note on plane hyperbolic geometry [same Glasnik **5**, 57–120 (1950); these Rev. **13**, 153].

F. A. Behrend (Melbourne).

Čarin, V. S. On a method of physical interpretation of Lobačevskiĭ's geometry. Uspehi Matem. Nauk (N.S.) **7**, no. 6(52), 207–208 (1952). (Russian)

A far-fetched interpretation, suggested by special relativity. The inertial systems play the role of points, and two segments AB and $A'B'$ are congruent if B' moves relatively to A' with the same velocity as B with respect to A.

H. Busemann (Los Angeles, Calif.).

Szász, Paul. Neue Bestimmung des Parallelwinkels in der hyperbolischen Ebene mit den klassischen Hilfsmitteln. Acta Sci. Math. Szeged **14**, 247–251 (1952).

The author gives a new proof for the classical formula $\cot \frac{1}{2}\tau = e^{t/k}$ which connects a hyperbolic distance t and its angle of parallelism τ. It is based on the principle that, if AA' and BB' are common diameters of two horocyclic arcs AB and $A'B'$, then the distance $AA'=BB'$ is proportional to the logarithm of the ratio of the arcs. This follows almost immediately from the possibility of transforming one horocycle into the other by a translation [see, e.g., the reviewer's Non-Euclidean geometry, Univ. of Toronto Press, 1st ed., 1942, 2nd ed., 1947, p. 244; these Rev. **4**, 50].

H. S. M. Coxeter (Toronto, Ont.).

Gordevskiĭ, D. Z. The classification of duality principles and of Desargues configurations in a multidimensional projective space. Učenye Zapiski Har'kov. Gos. Univ. **28**, Zapiski Naučno-Issled. Inst. Mat. Meh. i Har'kov. Mat. Obšč. (4) **20**, 155–161 (1950). (Russian)

The empty set, points, straight lines, $\cdots$, of a projective space are respectively called (-1)-element, 0-elements, 1-elements, $\cdots$. A "situation" $C_{k,l}$ (resp., a "manifold" $M_{k,l}$, $1 \leqq i < l-k$) is the set of all $(k+1)$-elements, $(k+2)$-elements, $\cdots$, $(l-1)$-elements (resp., all $(k+i)$-elements) incident to a given k-element and a given l-element incident to each other. Each $C_{k,l}$ (structurally isomorphic with a projective $(l-k-1)$-space) has its own duality principle (if $l \geqq k+2$). A lower (resp., upper) Desargue configuration $\underline{DK}_{k,k+l}$ (resp., $\overline{DK}_{k,k+l}$) in a $C_{k,k+l}$ is a set of $l+1$ $(k+1)$-elements (resp. $(k+l-1)$-elements) no l of which are incident to a $(k+l-1)$-element (resp., a $(k+1)$-element) of the $C_{k,k+l}$. Let there be given in a projective n-space $n+2$ hyperplanes forming a $\overline{DK}_{-1,n}$; each one of these hyperplanes is intersected by the $n+1$ others along a $\overline{DK}_{-1,n-1}$; the set of all the $\overline{DK}_{-1,n-i}$ obtained from the given $\overline{DK}_{-1,n}$ by repeating this process is called a "complete Desargues configuration". A few elementary enumerative results are given; a "generalized Desargues theorem" is proved; "flat Desargues configurations" are mentioned. *J. L. Tits.*

*Baer, Reinhold. **Linear algebra and projective geometry.** Academic Press Inc., New York, N. Y., 1952. viii+318 pp. $6.50.

This book has been written "to establish the essential structural identity of projective geometry and linear algebra". In the geometry of the 19th century the corresponding topic was called the antithesis of synthetic and analytic geometry. A comparison of the book under review with G. Fano's report on the antithesis [Encykl. Math. Wiss., Bd. III1, Heft 2, Teubner, Leipzig, 1907, pp. 221–288] shows the huge change of viewpoint and of methods in the last half century. A representation of the results of the new approach to the subject was due. A "projective geometry" is a complete complemented modular lattice of "points" (elements covering zero), "lines", etc., which satisfies three more structural conditions (see below); a "linear manifold"

(F, A) is a module A invariant for the operators of a (skew-) field F. Whereas a unified theory of the classical cases (F commutative, in particular, real or complex and A of finite rank) affords artificial restrictions, the modern treatment discloses the full correspondence between the geometries of dimension >2 and the linear manifolds. Besides the demonstration of its "Leitmotiv", the work contains a host of information (much of it unpublished before) on topics closely connected with the principal object and selected in a way which gives a uniform style to the book. At some places the author gives hints on related problems and quotes literature for further study; otherwise quotations are noted chapter by chapter. A list of works recommended for reference at the end of the book does not claim to be a complete bibliography.

Ch. I (Motivation) is introductory. Ch. II, The basic properties of a linear manifold. The lattice of the subspaces of (F, A) is discussed (inclusion, intersection, Dedekind's law, quotient spaces, laws of isomorphism and of complementation). Throughout the book, the elements of F are left operators, endomorphisms and isomorphisms are right operators. Hence for $0 \neq a \in A$, Fa is a point, $Fa+Fb$ (for $b \in A$, b non-$\in Fa$) is a line. (F, A) has a basis and its rank $r(A)$ is an invariant which, for given F, characterises (F, A) uniquely in the sense of isomorphism. The well-known rank-formulae of module theory hold for subspaces of finite rank only. Similarly, if C is a set of power c and F is a field of power d, the mappings of C into F form a linear manifold $[F, C]$ and $r([F, C]) = c$ or d^c according as c is finite or infinite. The adjoint space $L(A)$ of (F, A) consisting of the single-valued mappings of A into F (linear forms) and its Galois-relation with A is discussed. The classical reciprocity of A and $L(A)$ and the natural isomorphism of A and $L^2(A)$ hold if and only if $r(A)$ is finite; otherwise $r(L(A)) = d^{r(A)}$. The mappings of F into A (linear antiforms) form a linear manifold, the adjunct space, which is isomorphic to A. App. I, Systems of linear homogeneous equations. App. II, Paired spaces, i.e., spaces paired by a scalar product, as, e.g., A and $L(A)$. App. III, Fano's postulate: The 3 diagonal points of a quadrangle are non-collinear. This postulate is equivalent with the exclusion of 2 as characteristic of F.

Ch. III, Projectivities. These are inclusion-invariant isomorphisms of the subspaces of (F, A) on those of (G, B). If $r(A) > 2$, every projectivity is induced by a semilinear transformation of A (1st fundamental theorem). The auto-projectivities form a group which is homomorphic to the group of the semilinear transformations. The centraliser of the kernel of this homomorphism is the group T of the linear transformations. The auto-projectivities induced by T are called collineations; they are also induced by those semilinear transformations which are composed of a linear transformation of A and an inner automorphism of F. An auto-projectivity which leaves invariant all the subspaces of some hyperplane H $(r(A/H) = 1)$ is called a perspectivity. Let $r(A) = n$ be finite; then T is identical with the group generated by the perspectivities. Moreover, the mapping of a "simplex" of $n+1$ independent ordered points onto another simplex determines a collineation uniquely if and only if F is commutative (2nd fundamental theorem). The cross-ratio of 4 ordered collinear points consists of a full set of conjugate elements of F. Cross-ratios remain invariant for collineations. The permutations of the points on a line which induce semi-automorphisms of the cross-ratios are investigated. In App. I the homothetic group is constructed projectively. Of the points R, S, T, R', S', T', the first half is supposed to form a triangle which intersects a straight line in the three other points. Such a "basis sextet" being fixed, it can be completed by an "admissible quartet" to a (3-dimensional) Desargues configuration. If in ordinary geometry R', S', T' are points at infinity, the positions of the admissible quartet correspond $(1, 1)$ to the elements of the homothetic group. By a subtle investigation, the author shows that the same idea leads to a representation of the homothetic group in the general case. App. II concerns the connection of Pappus' theorem with the commutativity of F. App. III shows how the points on a line can be ordered when F is an ordered (skew-)field.

Ch. IV, Dualities. These are $(1, 1)$ mappings of the subspaces reverting the order produced by inclusion. As the subspaces form a well-ordered lattice, existence of a duality implies finiteness of the rank. This assumption is made throughout this chapter and the trivial case of rank 1 is excluded. (F, A) and (G, B) are dual whenever F is anti-isomorphic to G. Hence (F, A) admits an auto-duality whenever F affords an anti-automorphism α. In the classical case, auto-dualities (correlations) are determined by bilinear forms $f(x, y) = 0$, $x, y \in A$. Here the bilinear forms are generalised to semi-bilinear forms satisfying $f(tx, y) = tf(x, y)$; $f(x, ty) = f(x, y)t^\alpha$. An auto-duality the square of which is the identity is called polarity. For a polarity, either $f(x, x) = 0$ (nullsystem), or $f(x, y) = f(y, x)^\alpha$ where α is involutorial. Nullsystems exist in (F, A) if and only if F is commutative and $r(A)$ is even (classical case). A subspace W of A is called an N-subspace if $f(x, x) = 0$ for $x \in W$. Hence the points of a conic are the N-points of the generating polarity, etc. A subspace B is called strictly isotropic, isotropic, or non-isotropic for the polarity σ according as $B \subseteq B^\sigma$, $B \cap B^\sigma \neq 0$, $B \cap B^\sigma = 0$ holds. These subspaces are all thoroughly investigated, especially those with maximal property. Interesting rank-formulae as well as a σ-decomposition of A are deduced therefrom. The projectivities of A which are commutative with a polarity π, form the group of π. App. I, Sylvester's theorem of inertia. Let α be an involutorial anti-isomorphism of F, then we may construct a system of α-positivity $P \subset F$. Suppose π is a polarity represented by $f(x, y) = f(y, x)^\alpha$ and $p_1, \cdots, p_k \in A$ is a maximal system of points such that $f(p_i, p_j) = 0$ if and only if $i \neq j$. The following statement is a generalisation of Sylvester's theorem. If $f(p_i, p_i) > 0$ for $i < q$ and < 0 otherwise, then the pair of integers $q, k-q$ is independent of the choice of $p_1, \cdots, p_k$. App. II, Projective relations between lines induced by polarities. App. III, Let the characteristic of F be $\neq 2$ and σ be a polarity; then every hexagon of N-points in a plane satisfies Pascal's theorem if and only if σ can be represented by a symmetrical bilinear form. Pascal's theorem holds for every σ if and only if F is Euclidean. App. IV, The polarities with transitive group. These are generalisations of the symplectic and the unitary groups. The problems are closely connected with free mobility.

Ch. V, The ring of a linear manifold, concerns the ring P of the endomorphisms of (F, A). To every subspace S of A, there corresponds a left ideal $Q(S)$ of endomorphisms which map A into S (left annulet) and a right ideal (right annulet) $N(S)$ of annihilators of S. The relations between S and its annulets are reciprocal Galois dualities; the same holds for the relation of $N(S)$ and $Q(S)$ since these annulets annihilate one another. This "three-cornered Galois theory" is the basis of the investigation. The annulets are shown to be the ideals generated by a finite number of idempotents. The semilinear transformations of (F, A) upon (G, B) correspond $(1, 1)$ with the isomorphisms of $P(F, A)$ upon $P(G, B)$. Similarly for the auto-dualities of A and the anti-

automorphisms of $P(F, A)$. Appendix: The two-sided ideals in $P(F, A)$ correspond $(1, 1)$ with the cardinal numbers $\leqq r(A)$.

Ch. VI, The groups of a linear manifold. The manifolds are supposed to have a rank >2 and the characteristic of the fields to be $\neq 2$. When Λ and T denote the groups of the semilinear and of the linear transformations respectively, the following three statements are equivalent: $\Lambda(F, A)$ and $\Lambda(G, B)$ are isomorphic. $T(F, A)$ and $T(G, B)$ are isomorphic. (F, A) and (G, B) are either projective or dual each to the other. The isomorphisms $T(F, A)$, $T(G, B)$ admit a unique representation as products of a singular automorphism and an induced isomorphism. Moreover, this chapter contains a thorough investigation of involutions and an appendix on groups of involutions.

Ch. VII, Internal characterisation of the system of subspaces. Here analytic geometry (F, A) is confronted with a synthetic geometry S which satisfies 9 axioms. The essence of axioms I–V is that S is a complete and complemented modular lattice; axiom VI postulates that every $U(\neq 0) \in S$ contains a "point"; axiom VII postulates that when a point is contained in the sum of a family of elements of S, it is also contained in the sum of a finite subfamily; axiom VIII postulates the existence of more than 2 points on each line and axiom IX is Desargues' theorem. This axiom is shown to be equivalent with the condition that S is a proper subgeometry (and, in particular, a hyperplane) of another geometry (imbedding theorem). As S may not contain a hyperplane, the author cannot adopt Hilbert's proof of the imbedding theorem; the method of his proof is closely connected with that of Ch. III, App. I. The imbedding theorem and subsequent investigations on the group of a hyperplane lead eventually to the Representation Theorem which establishes the isomorphism of synthetic and analytic projective geometries in the generalised sense. The last section provides a sketch of the fundamentals of affine geometry drawn on similar lines to the preceding representation of projective geometry. A synthetic affine geometry is a projective geometry when a hyperplane and its subspaces are omitted. These geometries are shown to be isomorphic to the analytic affine geometries $V(F, A)$, where $V(F, A)$ means the lattice of the F-flocks of A over F. An F-flock is a system $W \subset A$ which is closed for the operations $\sum f_i w_i$, $w_i \in W$, $f_i \in A$, $\sum f_i = 1$. Appendix S is a survey of those results of abstract set-theory which are applied in the book, i.e., about everything an algebraist must know of that subject.

Geometers of the older type may wonder about a book which neither mentions the favourite subjects of classical geometry nor uses any method of analysis, but nevertheless gives a deep insight into the background of geometry. The results are obtained by a skillful combination of general algebra, lattice theory, and abstract set-theory with methods of classical synthetic geometry. A great number of "remarks" inserted into the text provide a welcome help to the reader who studies this very valuable and interesting work.

F. W. Levi (Berlin).

***Bieberbach, Ludwig. Theorie der geometrischen Konstruktionen.** Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften. Mathematische Reihe, Band 13. Verlag Birkhäuser, Basel, 1952. viii+162 pp. 18.70 Swiss francs.

The book is encyclopedic regarding constructions in the Euclidean plane, sketches briefly those on the sphere, and does not touch the hyperbolic plane. It begins with constructions by ruler in the whole plane, and passes to a bounded part of the plane, with or without additional data. For instance, given a square, then the parallel and perpendicular through a given point to a given accessible or inaccessible line can be constructed by ruler alone.

The next topic is the theorem that all constructions feasible by ruler and compass can be accomplished by ruler alone if an arc of a circle and its center are given. Also, all these constructions are feasible by the compass alone, and by a ruler with two parallel edges at a fixed positive distance, where the latter may be used to pass either one or the two different edges through two distinct points. Then follow constructions by means of a ruler on which two points are marked and their axiomatic characterization from Hilbert's Foundations of Geometry, also similar characterization of other constructions.

That angles cannot be trisected is shown by proving that there is no construction for a 20° angle. The author is doubtless right in considering this superior to an impossibility proof for a general construction, which leaves the disconcerting possibility that each individual angle can be trisected. The long tirade against van der Waerden [Moderne Algebra, v. I, 1st ed., Springer, Berlin, 1930, pp. 182–183; 2nd ed., 1937, p. 194] for dwelling on the general construction is, however, not justified. The issue is not as new as the author seems to think; it occurs in every construction in non-euclidean geometry, and is absent in many euclidean constructions because similarity makes the general and special case often equivalent, e.g., in the squaring of the circle.

The constructions leading to equations of order 3 or 4 are discussed as are various tools (other than ruler and compass, and too numerous to be listed here) for accomplishing them. As an example we mention that two angle irons or squares suffice, and the rather deep investigation on the use of dividers. Approximate trisections, by Dürer, the "Schneidermeister" Kopf, and improvements by Finsler follow.

That the circle cannot be squared is shown by modifying a method of Siegel and Gelfond for more advanced transcendency proofs. This is probably motivated by the desire of introducing a new idea into the textbook literature. The proof is hardly as simple as proofs based on the older ideas; compare, for instance, the recent version of Steinberg and Redheffer [Pacific J. Math. **2**, 231–242 (1952); these Rev. **13**, 921]. The already mentioned sketch of spherical constructions and an appendix with references to the literature section by section conclude the book.

The style is clear and lively. The references are, wherever possible, to the author's own books, which is unfortunate in cases where other books are better known and more easily available.

H. Busemann (Los Angeles, Calif.).

***Hjelmslev, Johannes. On the general foundations of geometry.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 3–12. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr. (Danish)

A very brief account of the foundations of geometry from antiquity to Hilbert, followed by a somewhat more detailed discussion of the basic ideas in the author's papers [Danske Vid. Selsk. Math.-Fys. Medd. **8**, no. 11; **10**, no. 1 (1929); **19**, no. 12 (1942); **22**, nos. 6, 13 (1945); **25**, no. 10 (1949); these Rev. **7**, 472, 473; **8**, 83; **11**, 124].

H. Busemann (Los Angeles, Calif.).

Buck, R. C. Unique segments in metric spaces. Amer. Math. Monthly **60**, 100–102 (1953).

The note shows that if the six mutual distances of each quadruple of elements x, y, p, q of a metric space satisfy the inequality $xy^2+pq^2 \leqq xp^2+yq^2+2xq \cdot yp$, then no two distinct points of the space are joined by more than one metric segment. *L. M. Blumenthal* (Columbia, Mo.).

Hopf, H. Über Zusammenhänge zwischen Topologie und Metrik im Rahmen der elementaren Geometrie. Math.-Phys. Semesterber. **3**, 16–29 (1953).

Expository lecture.

Convex Domains, Extremal Problems, Integral Geometry

Rankin, R. A. A problem concerning three-dimensional convex bodies. Proc. Cambridge Philos. Soc. **49**, 44–53 (1953).

Let K be a convex 3-dimensional body symmetrical in the origin O and let $\mu \geqq 1$ be given. It is shown that there is a plane π through O not containing the z-axis and a linear transformation T such that: (a) T leaves each point on the z-axis invariant; (b) T leaves invariant every direction in π; (c) TK is contained in the body μK formed by expanding K by the factor μ; and (d) the determinant of T is not less than $[\frac{1}{4}(1+3\mu)]^2$. The result is shown to be best possible; it is required for the proof of a result of the author on the anomaly of a convex body [Proc. Cambridge Philos. Soc. **49**, 54–58 (1953); these Rev. **14**, 624]. *C. A. Rogers.*

Hammer, Preston C. The centroid of a convex body. Proc. Amer. Math. Soc. **2**, 522–525 (1951).

Soit C un corps convexe plan, x un point intérieur, yz une corde passant par x (y et z sur la frontière de C); on considère le rapport de longueurs: yx/yz et on pose, lorsque la corde varie: $r(x)=\max(yx/yz)$, puis: $r(C)=\min r(x)$. Démonstration simple, par utilisation du barycentre, du résultat de B. H. Neumann [J. London. Math. Soc. **14**, 262–272 (1939); ces Rev. **1**, 158] selon lequel: $1/2 \leqq r(C) \leqq 2/3$, et de sa généralisation au cas de n dimensions:

$$1/2 \leqq r(C) \leqq n/(n+1).$$

J. Favard (Grenoble).

Hammer, Preston C. Convex bodies associated with a convex body. Proc. Amer. Math. Soc. **2**, 781–793 (1951).

(Notations de l'analyse ci-dessus.) y désignant un point de la frontière C (plan!), on considère les corps convexes $C_y(r)$ homothétiques de C par rapport à y et dans le rapport r (>0); l'auteur étudie les ensembles:

$$\begin{aligned} C(r) &= \bigcap_y C_y(r) \quad \text{pour} \quad r<1, \\ &= \bigcup_y C_y(r) \quad \text{pour} \quad r>1, \\ &= C \quad \text{pour} \quad r=1, \end{aligned}$$

et donne diverses propriétés; en particulier, $C(r)$ est vide pour $r<r(C)$; $C[r(C)]$ se réduit à un seul point (point critique); les frontières des $C(r)$ sont convexes et couvrent simplement le plan; on peut passer de $C(r)$ à C par l'opération inverse de la précédente pour $r>1$, mais cela n'est pas toujours possible pour $r<1$. *J. Favard* (Grenoble).

Klee, V. L., Jr. The critical set of a convex body. Amer. J. Math. **75**, 178–188 (1953).

(Notations des notes de Hammer reviewées ci-dessus.) Pour un corps convexe C de l'espace à n dimensions, l'ensemble C^* où $r(x)=r(C)$ (formé d'un seul point pour $n=2$) est convexe, et l'auteur démontre que:

$$r(C)/[1-r(C)]+\dim C^* \leqq n;$$

que, pour $x \in C^*$, l'ensemble des points y de la frontière de C pour lesquels $yx/yz=r(C)$ n'est contenu dans aucun demi-espace ouvert ne contenant pas C^*; que si $r(C)=n/(n+1)$, C est un simplexe [résultat annoncé par W. Süss, Arch. Math. **1**, 127–128 (1948); ces Rev. **10**, 395]. Il généralise enfin le résultat classique de Helly sur l'intersection des ensembles convexes. *J. Favard* (Grenoble).

Lorch, E. R. Differentiable inequalities and the theory of convex bodies. Trans. Amer. Math. Soc. **71**, 243–266 (1951).

Soit, dans R^n, Ω un espace de rayons, ne contenant pas l'origine, ouvert, convexe, contenant avec x le point cx ($c>0$); $\phi(x)=\phi(x_1, \cdots, x_n)$ une fonction définie dans Ω, positive et positivement homogène de degré 1, admettant des dérivées partielles continues jusqu'à un ordre suffisant; on pose: $G(x)=r^{-1}\phi^r$ ($r \neq 0$); $G(x)=c$, avec $rc>0$, représente une famille de surfaces homothétiques que ne dépend que de ϕ.

La transformation: $x_i^*=\partial G/\partial x_i$ fait correspondre à x un point x^* d'un espace de rayons Ω^* (pas nécessairement ouvert cependent) ($r \neq 1$), et on suppose que

$$(1) \qquad (r-1)\sum G_{ij}(x)z_iz_j \quad \left(G_{ij}=\frac{\partial^2 G}{\partial x_i \partial x_j}\right)$$

n'est pas singulière dans Ω(dét $|G_{ij}| \neq 0$); on a alors un homéomorphisme local entre Ω et Ω^*, qui est global sous certaines conditions précisées; à $G(x)=c$ correspond la famille adjointe $F(x^*)=b$ (F positivement homogène de degré r': $1/r+1/r'=1$). Si la forme (1) est définie positive pour tout $x \in \Omega$, et si $r>1$, les surfaces $G=c$ sont dites convexes; si $r<1$, elles sont dites concaves. Si la réunion de Ω et de l'origine est convexe, on a alors: $\sum x_i^* y_i \leqq \psi(x^*)\phi(y)$, ψ est l'adjointe de ϕ, le cas d'égalité est précisé.

L'auteur introduit les classes naturelles de surfaces convexes pour lesquelles r, Ω, puis Ω^* sont les mêmes, qui sont toutes convexes ou concaves, la classe étant fermée relativement à la multiplication par un scalaire et à l'addition (si $G^0=c^0$, $G^1=c^1$ appartiennent à la classe, $\lambda^0 G^0+\lambda^1 G^1=c$ appartient à la classe pour λ^0, $\lambda^1>0$). On peut introduire une distance entre les surfaces d'une classe, puis compléter l'espace; on arrive ainsi à montrer que les surfaces convexes au sens ordinaire peuvent être approchées par des surfaces analytiques convexes dans le sens de l'auteur. Le travail s'achève par des formules relatives aux volumes mixtes entre lesquels l'auteur espère pouvoir, par sa méthode, trouver des inégalités nouvelles. *J. Favard.*

Kubota, Tadahiko. Einige Ungleichheiten für die Eilinien und Eiflächen. Proc. Japan Acad. **24**, no. 7–8, 1–3 (1948).

Let C be a closed convex curve in the plane and F, L, D its area, length, and diameter respectively. If M is the mixed area of C and the curve obtained from C by revolution through 180°, then $DL \geqq 2(F+M)$ with equality only for curves with constant width. A consequence is Hayashi's inequality $DL \geqq 4F$ with equality only for the circles. The

latter fact is extended to space: If V, O, D are volume, area, and diameter of a convex surface, then $DO \geqq 6V$ with equality only for spheres. *H. Busemann.*

Süss, W. Eine elementare kennzeichnende Eigenschaft des Ellipsoids. Math.-Phys. Semesterber. **3**, 57–58 (1953).

It is proved in an entirely elementary way that an ovaloid, all of whose orthogonal projections are ellipses, is an ellipsoid. *S. B. Jackson* (College Park, Md.).

Pogorelov, A. V. Regularity of a convex surface with given Gaussian curvature. Mat. Sbornik N.S. **31(73)**, 88–103 (1952). (Russian)

For a convex surface C the ratio of the area of a set to that of its spherical image may converge when the former contracts to a point p, although C possesses at p no second derivatives: e.g., $2z=x^2\log z+y^2\log^{-1}z$ at $x=y=0$. Define this limit, if it exists, as Gauss curvature. If the Gauss curvature exists at every point of C and is positive and finite, then C is differentiable and its spherical mapping is one-to-one [A. D. Alexandroff, C. R. (Doklady) Acad. Sci. URSS (N.S.) **36**, 195–199 (1942); these Rev. **4**, 169]. A point of C is therefore determined by the coordinates u, v of its spherical image. If u, v are analytical coordinates on the unit sphere, the present paper proves: If the Gauss curvature $K(u, v)$ of C at (u, v) is positive and m times differentiable, $m \geqq 3$, then C is at least $m+1$ times differentiable as surface in E^3. *H. Busemann* (Los Angeles, Calif.).

Hadwiger, H. Zur isoperimetrischen Ungleichung für k-dimensionale konvexe Polyeder. Nagoya Math. J. **5**, 39–44 (1953).

Let V be the volume and F the surface area of a k-dimensional convex polytope bounded by n cells. Consider n cones of revolution the basic spheres of which have a total area equal to F and the sum of the solid angles at the vertices of which equals the surface area of the k-dimensional unit sphere. It is proved that the total volume of these cones is $\geqq V$ and equality holds only for a 2-dimensional regular polygon. This involves a sharpening of the classical isoperimetric inequality. The proof rests on ideas of Goldberg [Tôhoku Math. J. **40**, 226–236 (1935)] who proved the above theorem for $k=3$. *L. Fejes Tóth* (Veszprém).

Pikus, D. L. The isoperimetric problem in the Lobačevskiĭ plane. Trudy Sem. Vektor. Tenzor. Analizu **9**, 456–461 (1952). (Russian)

Without reference to the literature the isoperimetric problem in the hyperbolic plane is solved by adapting Steiner's "Viergelenkverfahren" to hyperbolic geometry. *H. Busemann* (Los Angeles, Calif.).

Eggleston, H. G. On closest packing by equilateral triangles. Proc. Cambridge Philos. Soc. **49**, 26–30 (1953).

Let us place into a given convex domain an infinite set of non-overlapping homothetic convex discs. An arrangement of the discs inside the domain for which the set S of uncovered points has the least fractional dimension in the sense of Hausdorff is said to be a close-packing. The author considers close-packings of equilateral triangles inside an equilateral triangle of unit side-length of opposite orientation and shows that S has dimension $\log 3/\log 2$ and a measure $\geqq 1/3$ and $\leqq 1$ in this dimension. *L. Fejes Tóth* (Veszprém).

Thüring, R. Studien über den Holditchschen Satz. Verh. Naturforsch. Ges. Basel **63**, 221–252 (1952).

Eine Gerade kehrt nach einer Bewegung in der Ebene in die Anfangslage zurück; drei ihrer Punkte A, B und C beschreiben Kurven mit den Flächeninhalten (A), (B), (C). Ist B-C die Entfernung von B und C u.s.w., dann sagt der Satz von Holditch (1858);

$$(B\text{-}C)\cdot(A)+(C\text{-}A)\cdot(B)+(A\text{-}B)\cdot C+n\pi\cdot(A\text{-}B)(B\text{-}C)(C\text{-}A)=0.$$

Verf. betrachtet den Fall dass A und B dasselbe Oval R (eine geschlossene doppelpunktfreie konvexe Kurve) durchlaufen und $n=1$ ist. Ist L die Länge der Sehne AB, dann ist diese Holditchsche Bewegung dann und nur dann möglich wenn $L \leqq \Delta$, wo Δ die Minimalbreite von R ist. Die Bewegung ist im allgemeinen eindeutig (d.h. jeder Richtung entspricht eine und nur eine Sehnenlage); eine Ausnahme bilden nur die Bewegungen wobei die Sehne längs eines geradlinigen Randstückes von R gleitet. Hinreichend für die Doppelpunktfreiheit der von C beschriebene Kurve ist: $L<\Delta$ wenn C ausserhalb der Strecke AB liegt, $L<2r$ wenn C zwischen A und B liegt; hier ist $2r$ der maximale Durchmesser eines Inkreises von R. Die Bewegung heisst monoton wenn die von A und B bescriebenen Bogenlängen eigentlich monoton wachsen; hinreichend für die Monotonie ist $L<N$, notwendig ist $L \leqq N$; hier ist N die untere Grenze der Normallängen von R. Verf. betrachtet noch speziell Ovale mit durchwegs stetiger Krümmung und gibt für diesen Fall eine weitere hinreichende Bedingung für die Doppelpunktfreiheit von R, in der der maximale und der minimale Krümmungskreis von R vorkommen. Er diskutiert am Schluss einige Beispiele wobei R bzw. ist ein gleichseitiges Dreieck, die Reuleauxsche Kurve konstanter Breite, eine Ellipse und eine Parabel. *O. Bottema.*

Biernacki, M. Sur quelques propriétés des fonctions de distances. J. Math. Pures Appl. (9) **31**, 305–318 (1952).

The starting point is a theorem of Aumann [S.-B. Preuss. Akad. Wiss. **1933**, 924–931] refined by H. Kneser [ibid. **1934**, 426–431]. In the present paper theorems of similar kind are obtained, for instance the following two. Let A_i be arbitrary points in space, E a closed set. We have then

$$\sum_{i=1}^{n} \max MA_i \leqq 2 \max\left(\sum_{i=1}^{n} MA_i\right),$$

$$\prod_{i=1}^{n} \max MA_i \leqq [8(1+2^{1/2})]^n \max \prod_{i=1}^{n} MA_i.$$

The constant 2 in the first inequality is the best possible. This is not the case with the constant $8(1+2^{1/2})$ occurring in the second inequality. *G. Szegő* (Stanford, Calif.).

Vidal Abascal, E. On the fundamentals of integral geometry. Revista Mat. Hisp.-Amer. (4) **12**, 290–310 (1952). (Spanish)

Expository paper.

Algebraic Geometry

Rozenfel'd, B. A., and Skopec, Z. A. Quadratic Cremona transformations in the plane and complex numbers. Doklady Akad. Nauk SSSR (N.S.) **83**, 801–804 (1952). (Russian)

Es ist bekannt, dass man die quadratischen Transformationen der projektiven Ebene E mit einem reellen und 2

konjugiert komplexen Fundamentalpunkten auch als gebrochene lineare Transformationen schreiben kann, wenn man nur E in geeigneter Weise als komplexe Zahlenebene umdeutet. Verfasser zeigt, dass man auch die anderen Typen reeller quadratischen Verwandtschaften in E als gebrochene lineare Transformationen in einer geeigneten Algebra schreiben kann. Als solche tritt bei den Transformationen mit 3 reellen getrennten Fixpunkten das System $z=a+be$ ($e^2=+1$, a, b reell) und bei denen mit zusammenfallenden Fixpunkten das System $z=a+b\epsilon$ ($\epsilon^2=0$) auf. Analog wie beim ersten Fall lassen sich dann auch diese Abbildungen als Produkte geeigneter Inversionen erklären. *W. Burau.*

Purcell, Edwin J. Further Cremona involutions in $[n]$. Univ. Nac. Tucumán. Revista A. **8**, 19–29 (1951).

A Cremona involution in n-dimensions may have a locus of fundamental points of the first kind (basis points of the systems of primals defining the involution) consisting of a single variety. The author classifies such Cremona involutions in n dimensions and treats two types in detail.

T. R. Hollcroft (Aurora, N. Y.).

Muracchini, Luigi. Sulle trasformazioni cremoniane che conservano le aree od i volumi. Boll. Un. Mat. Ital. (3) **7**, 388–392 (1952).

The author finds a necessary and sufficient condition that a Cremona transformation between two affine planes preserve area. The basic result is as follows: A necessary and sufficient condition that a Cremona transformation between two affine planes be such that corresponding areas are in a constant ratio is that (i) the lines at infinity correspond and (ii) the Jacobian curve (in either plane) reduce to a curve Γ counted three times, and Γ together with the line at infinity be a curve of the corresponding homaloidal net. The author makes the interesting observation that the group of area-preserving Cremona transformations is not generated by the quadratic transformations belonging to it. There are (elementary) extensions to space of higher dimension.

J. A. Todd (Cambridge, England).

Manara, Carlo Felice. Identità birazionale dei piani tripli aventi una stessa curva di diramazione. Atti Sem. Mat. Fis. Univ. Modena **5**, 54–65 (1951).

The author proves that a triple plane is uniquely determined, to within birational equivalence, by its branch curve.

J. A. Todd (Cambridge, England).

***Villa, M. Transformations ponctuelles et transformations crémoniennes.** Deuxième Colloque de Géométrie Algébrique, Liège, 1952, pp. 41–68. Georges Thone, Liège; Masson & Cie, Paris, 1952. 375 Belgian francs; 2625 French francs.

An expository article dealing in the main with approximations to general point-point transformations between linear spaces by means of Cremona transformations.

J. A. Todd (Cambridge, England).

Villa, Mario. Varietà quasi-asintotiche e trasformazioni puntuali. Ann. Univ. Ferrara. Sez. VII. (N.S.) **1**, 17–21 (1952).

The author points out that many important properties of point-point correspondences between two linear spaces give rise to relations of a quasi-asymptotic character when interpreted on the corresponding Segre variety. The case of transformations between two planes serves to provide his illustrative examples. *J. A. Todd.*

Vaona, Guido. Sulle curve di una varietà quasi-asintotica. Boll. Un. Mat. Ital. (3) **7**, 411–420 (1952).

The author studies curves lying on a V_h, which is quasi-asymptotic for a V_m, which are quasi-asymptotic both for V_m and for the pair (V_h, V_m). The problems considered concern the existence of such curves, and the number of parameters on which they depend. *J. A. Todd.*

Rosina, B. A. Alcune osservazioni sulle coniche generalizzate. Ann. Univ. Ferrara Sez. VII. (N.S.) **1**, 135–142 (1952).

Piazzolla-Beloch [Ann. Univ. Ferrara. Parte I. **6**, 91–101 (1947); these Rev. **13**, 155] defined a generalized conic as a curve of order $2n$ with two n-fold points at infinity, and showed that various properties of these curves are analogous to those of conics. In the present paper, a more general curve type, also called a generalized conic, is treated. A curve of this type is of order $2n$ ($n>1$) with two points, each of multiplicity at least two, at infinity, each point counting as n intersections of the curve with the line at infinity. These curves have the same diametral properties as the generalized conics defined by Piazzolla-Beloch. *T. R. Hollcroft.*

Nakai, Yoshikazu. On the genus of algebraic curves. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **27**, 163–165 (1952).

An algebraic proof of the theorem that, if C is a curve without multiple points belonging to an irreducible system Σ of curves on an algebraic variety which is without multiple points, then the genus of C is equal to the genus of a generic member of Σ.

W. V. D. Hodge (Cambridge, England).

***Chisini, O. Courbes de diramation des plans multiples et tresses algébriques.** Deuxième Colloque de Géométrie Algébrique, Liège, 1952, pp. 11–27. Georges Thone, Liège; Masson & Cie, Paris, 1952. 375 Belgian francs; 2625 French francs.

Cette communication fait le point des résultats obtenus récemment pour la caractérisation des courbes planes diramantes d'un plan multiple. La caractérisation algébrique par la construction de la fonction $z(x, y)$ a été étendue au cas d'un plan multiple équivalent à une surface projetée d'un point multiple unique, ceci par l'extension du théorème d'Halphen caractérisant la projection de l'intersection complète de deux surfaces. La caractérisation topologique à partir de la condition d'Enriques a fait de grands progrès, depuis l'introduction par l'A. de la tresse représentative de la courbe diramante; l'étude des propriétés de la tresse par des transformations de l'ordre des lacets a permis dans certains cas de construire une véritable algèbre; les considérations de variation continue de la courbe permises par la condition d'Enriques facilitent souvent la construction des tresses, et permettent de déterminer les plans multiples distincts, ayant la même courbe de diramation, par le choix convenable des échanges de déterminations possibles.

Les résultats obtenus sont encourageants, mais les applications n'ont pas encore permis de dépasser les plans multiples et les courbes de diramation dont les ordres restent de très petits nombres.

B. d'Orgeval (Alger).

*Nollet, L. **Introduction des courbes quasi irréductibles d'une surface algébrique. Application à la régularité de certains systèmes linéaires.** Deuxième Colloque de Géométrie Algébrique, Liège, 1952, pp. 211–223. Georges Thone, Liège; Masson & Cie, Paris, 1952. 375 Belgian francs; 2625 French francs.

Conférence faite par l'auteur au deuxième Colloque de géométrie algébrique. Dans l'introduction l'auteur (en s'efforçant, dans la mesure du possible, d'établir un pont entre les différents vocabulaires utilisés dans le Colloque de Liège) rappelle quelques notions fondamentales de la géométrie algébrique: définition de variété algébrique simple sur un corps k commutatif et défini à un isomorphisme près; variétés algébriques irréductibles et réductibles; systèmes algébriques; intersection et interférence de variétés algébriques; multiplicité d'intersection; variétés de Weil et de Gröbner; etc. Etant alors donnée sur le corps complexe une surface algébrique irréductible F sans points multiples, on dira qu'une courbe C de F est quasi-irréductible si $(C, X) \geqq 0$ pour toute courbe positive X de F, où (C, X) signifie la somme des multiplicités d'intersection des points communs à C et X. L'intérêt de cette notion dépend surtout du fait que, si F est dépourvue de courbes exceptionnelles de première espèce et de faisceaux de courbes unicursales, toute courbe canonique de F est quasi-irréductible, et donc aussi les courbes pluricanoniques, paracanoniques, pseudocanoniques, para- et pseudopluricanoniques de F. A l'aide du théorème de Riemann-Roch on déduit plusieurs applications. On trouve, par exemple, que tout système linéaire pseudo-v-canonique pour $v \geqq 3$ est régulier. (Le cas $v=2$ est plus difficile à traiter; la méthode précédente s'applique toutefois encore dans un certain nombre de cas.)

E. G. Togliatti (Gênes).

Alguneid, A. R. Degeneration of space collineations. Proc. Egyptian Acad. Sci. **7** (1951), 1–17 (1952).

Van der Waerden [Math. Ann. **115**, 645–655 (1938)] introduced the notion of a complete conic and Semple [J. London Math. Soc. **23**, 258–267 (1948); these Rev. **10**, 472] that of a complete quadric, the latter being the quadratic variety in three-dimensional space considered simultaneously as a point locus, as an envelope of planes, and as a complex of tangent lines. The author defines a complete collineation in space as the combination of the point, plane, and line transformation and characterizes it by means of algebraic forms. The interesting part is as in the former cases the study of the degenerations. There are seven types of them and the author, representing the complete collineations on the points of a manifold W_{15} proves that the image of the degenerations of each type is free from singularities. There are three 14-dimensional manifolds ψ_1, ψ_2, ψ_3, three 13-dimensional ones $\psi_2\psi_3$, $\psi_3\psi_1$, $\psi_1\psi_2$ and one 12-dimensional $\psi_1\psi_2\psi_3$.

O. Bottema (Delft).

Conforto, Fabio, e Gherardelli, Francesco. Classificazione delle superficie ellittiche con un fascio ellittico di curve di genere tre. Ann. Mat. Pura Appl. (4) **33**, 273–351 (1952).

In this memoir, the authors give a remarkable contribution to the classification of elliptic surfaces. It is known that any elliptic surface Φ has a group Γ of ∞^1 birational transformations into itself; therefore on the surface there is a pencil of elliptic curves $\{K\}$, whose genus is equal (Enriques) to the geometric genus p_g of Φ, the curves K being the paths (trajectories) of Γ. According to Enriques, the surface Φ possesses also an elliptic pencil $\{C\}$, which is transformed into itself by Γ. The surfaces Φ depend thus on the following characters, which are integers: 1) the geometric genus p_g; 2) the determinant n (the number of intersections of a curve C with a curve K; 3) the genus π $(\geqq 1)$ of the curves C; 4) the number t of the multiple curves K $(K \equiv sK_s)$ with their order of multiplicity s_i $(i=1, 2, \cdots, t)$. The above characters are connected by the relation (due to Zeuthen):

$$\pi = 1 + n(p_g - 1) + \sum_{i=1}^{t} \frac{s_i - 1}{2s_i} n. \tag{1}$$

The groups of points common to a curve C and to a curve K form on Φ an involution, which is generated by a finite group G_n of birational transformations of Φ into itself, G_n being either cyclic or abelian with base two. The general construction of the elliptic surfaces may be obtained using results by Enriques and Chisini. The complete classification of the elliptic surfaces had been given by Enriques for the case $\pi=1$, and by Campedelli for $\pi=2$, $p_g=0$. [See, also for the literature, F. Enriques, Le superficie algebriche, Zanichelli, Bologna, 1949, Chapters X, XI; these Rev. **11**, 202]. The authors give the complete classification of the elliptic surfaces for the case of $\pi=3$, which appears more difficult than the cases of $\pi=1$, 2, since a curve of genus three is not, in general, hyperelliptic. Moreover, the authors treat thoroughly, always under the assumption $\pi=3$, the case of positive p_g.

A first idea in the study of the possible types of surfaces with $\pi=3$ may be obtained by the arithmetic study of equation (1); in this way one finds 31 possible cases, a priori, for the sets of numbers p_g, n, t, s_i $(i=1, 2, \cdots, t)$, namely, 23 cases in which $p_g=0$, 6 for which $p_g=1$, and one for which $p_g=2$ or 3. Secondly, the authors determine all the possible types of curves of genus three, with groups of birational transformations into themselves, the groups being either cyclic or abelian with base two; for the non-hyperelliptic case, the authors use results of F. Klein, S. Kantor, A. Wiman, E. Ciani, on the self-projective plane quartics; for the hyperelliptic case, the question may be reduced to the determination of all the possible types of sets of eight points on a projective line, transformed into themselves by a finite group of projectivities. In this way one finds that only 21 of the possible 31 cases are geometrically possible. The (geometrically) possible values for the determinant n are, in particular, only $n=1, 2, 3, 4, 6, 7, 8, 9, 12, 14, 16$. The last part of the memoir gives then effective construction, following Enriques' ideas, of all possible types of elliptic surfaces. Some new facts turn out here in comparison with the cases $\pi=1, 2$. First of all, a given set of numbers p_g, n, t, s_i $(i=1, 2, \cdots, t)$ which fulfil (1) and is geometrically possible does not always determine a single continuum (possibly only one) of curves C, and, therefore, only one type of elliptic surface; e.g., for $p_g=0$, $n=7$, $t=3$, $s_1=s_2=s_3=7$, one obtains a first type of elliptic surfaces, on which the curves C are birationally equivalent to the hyperelliptic curve $y^2=x^7-1$, and a second type, on which the C's are equivalent to Klein's quartic (which is not hyperelliptic) $x^3y+y^3z+xz^3=0$. But, not even the distinction between hyperelliptic and non-hyperelliptic curves C is always sufficient; it is quite possible, in fact, that for a same given set of numbers p_g, n, t, s_i, there are two types of elliptic surfaces on which the C's are hyperelliptic, and one type on which they are not hyperelliptic. It is even possible that the surfaces of a certain type, on which the C's are hyperelliptic, are particular in a larger family for which the C's are not

hyperelliptic. Finally, if $p_g=1$, several classes of surfaces are possible which differ only for the different value of the modulus of the pencil $\{K\}$. All these facts prove therefore that the problem of the classification of the elliptic surfaces appears in its complete generality, and, one may say, with all its natural complications, only for the case of $\pi \geq 3$.

E. Martinelli (Genoa).

Godeaux, Lucien. Sur un faisceau de surfaces algébriques irrégulières. Bull. Soc. Roy. Sci. Liège **21**, 314–319 (1952).

En se rapportant à une communication faite au III. Congrès des mathématiciens autrichiens à Salzbourg en Septembre 1952, l'A. considère ici le faisceau de surfaces F d'ordre 12 répresenté par l'équation suivante: $\sum x_i^{12}-2\sum x_i^6x_j^6+ax_1^3x_2^3x_3^3x_4^3=0$ $(i, j=1, 2, 3, 4; i\neq j)$. La surface générale de ce faisceau est irréductible et possède 36 points doubles tacnodaux particuliers distribués six par six sur les arêtes du tétraèdre de réference; si P est un quelconque de ces points, la surface F possède une droite double p infiniment voisine de P et une droite double p' infiniment voisine de p. Les surfaces adjointes F' de F sont des surfaces d'ordre 8, passant par tous les 36 points P en y touchant la surface F. En écrivant directement leur équation, on trouve pour le genre géométrique de F la valeur 63, tandis que, les conditions imposées aux surfaces F' étant $36\times 3=108$, le genre arithmétique de F est 57. La surface F possède donc l'irrégularité 6. Le comportement des adjointes F' aux points P est controlé à l'aide d'une transformation quadratique générale.

E. G. Togliatti (Gênes).

***Gaeta, Federico. Quelques progrès récents dans la classification des variétés algébriques d'un espace projectif.** Deuxième Colloque de Géométrie Algébrique, Liège, 1952, pp. 145–183. Georges Thone, Liège; Masson & Cie, Paris, 1952. 375 Belgian francs; 2625 French francs.

This is an expository article, reporting on the author's theory of algebraic varieties of finite residual, and its connections with the theory of arithmetically normal varieties due to Zariski.

J. A. Todd (Cambridge, England).

Gaeta, Federico. Sui sistemi lineari appartenenti al prodotto di più varietà algebriche. Ann. Mat. Pura Appl. (4) **33**, 91–118 (1952).

Let V_i be a non-singular, irreducible variety of dimension d_i in $[r_i]$, $i=1, 2, \cdots, k$, let Σ_ρ be the Segre variety $\Pi[r_i]$, and let $W_i=\Pi V_i$ be represented on Σ_ρ. If $|L_i|$ denotes a (not necessarily complete) linear system of (d_i-1)-dimensional cycles on V_i, and if $W_i=\Pi(V_j, j\neq i)$, then the $(t-1)$-dimensional cycles $L_i\times W_i$ are distributed into k linear systems $|L_i\times W_i|$ on W_i. The minimal sum $|L|$ of these linear systems is a linear system which is defined to be the product $\Pi|L_i|$ of the systems $|L_i|$. The members of $|L|$ are connexions or pluricorrespondences of valence zero. The system $|L|$ is (a) complete, (b) free of base points, if and only if its factors have these properties. The dimension n of $|L|$ is related to the dimensions n_i of the factors by the formula, $n+1=\Pi(n_i+1)$. For any h, the h-fold of $|L|$ is the product in the above sense of the h-folds of $|L_i|$, and the postulation $\chi(h, W_i)=\Pi\chi(h, V_i)$. This last result is used to express relative invariants of W_i in terms of those of V_i. If K_i is the relative canonical system on V_i, and K that of W_i, it is indicated that $K=\Pi K_i$, where the product is intended in the above sense. It follows that $p_g(W_i)=\Pi p_g(V_i)$, where p_g is the relative geometric genus.

In the second part of the paper these results are applied to several special cases. In particular, products W_i are considered on which every $(t-1)$-dimensional connexion is of valence zero. For such W_i, the problem of finding minimal non-singular models reduces to the corresponding problem for the factors.

H. T. Muhly (Iowa City, Iowa).

Gröbner, Wolfgang. Über das arithmetische Geschlecht einer algebraischen Mannigfaltigkeit. Arch. Math. **3**, 351–359 (1952).

Let k be a field and let $x_0, x_1, \cdots, x_n$; $y_0, y_1, \cdots, y_m$ be indeterminates. A set of $m+1$ forms $\varphi_j(x)$ of like degree define a mapping T of the H-ideals A_x of $k[x]$ into the set of H-ideals A_y of $k[y]$ under which $A_y(=A_xT)$ is the set of forms $F\in k[y]$ such that $F(\varphi)\in A_x$. Such a mapping is said to be birational for a pair (A_x, A_yT) in case there is a transformation U, $A_y\to A_x$, such that $A_x=A_xTU$. The "Veronese transformation" V_σ defined by the forms $x_0^{i_0}x_1^{i_1}\cdots x_n^{i_n}$, $\sum i_j=\sigma$, is birational for every pair (A_x, A_xV_σ). Any transformation T is composed of a Veronese transformation, a linear transformation, and projections, where by a projection is meant a transformation

$$A_x\to A_xP=A_x\cap k[x_0, \cdots, x_{n-1}].$$

If the virtual arithmetic genus $p_a(A)$ of the ideal A is defined in the usual way by means of the Hilbert polynomial $H(A, n)$, the author observes that $p_a(AV_\sigma)=p_a(A)$, and under suitable restrictions on P, also $p_a(AP)=p_a(A)$.

H. T. Muhly (Iowa City, Iowa).

Kundert, E. G. A relation between poles and zeros of a simple meromorphic differential form and a calculation of Chern's characteristic classes of an algebraic variety. Proc. Nat. Acad. Sci. U. S. A. **38**, 893–895 (1952).

Cette note contient deux nouvelles applications des résultats obtenus par l'auteur dans Ann. of Math. (2) **54**, 215–246 (1951) [ces Rev. **13**, 374]. Soit α une forme différentielle méromorphe, de degré un, sur une variété complexe M^n et soit $C(\alpha)$ (resp. $S(\alpha)$) la classe de cohomologie duale de la classe d'homologie du cycle des zéros (resp. cycle polaire) de α; alors, si les Γ_k désignent les classes de Chern covariantes, $C(\alpha)$ peut être exprimée comme un polynôme de degré n en $S(\alpha)$ dans l'anneau de cohomologie de M^n: (1) $C(\alpha)=S^n(\alpha)+\sum_{i=1}^{n}\Gamma_{n-i+1}S^{n-i}(\alpha)$. En effet si, au voisinage d'un point de M^n, on désigne par p_i/q les coefficients de α et si F est le champ de directions covariantes $\{q, p_1, \cdots, p_n\}$, $C(\alpha)$ est l'obstruction de F. Si G est la champ de directions correspondant au champ de vecteurs covariants de Chern, le cocycle différence $d(F, G)$ détermine la classe $S(\alpha)$; la formule (1) résulte de la formule notée (3) dans l'analyse citée. Les classes de Chern Γ_k d'une variété algébrique V^n immergée dans l'espace projectif complexe P^{n+1} sont déterminées par les formules suivantes: Si Π_k est la classe de cohomologie duale de la classe d'homologie de $P^k\cap V^n$ où P^k est un k-plan de P^{n+1}:

$$(2)\qquad \begin{aligned}(-1)^n\Gamma_1&=\Pi_n{}^n+\sum_{k=1}^{n}\Pi_{n-k+1}\cdot\Pi_n{}^{n-k},\\ (-1)^{n-k}\Gamma_k&=\frac{1}{(k-1)!}\frac{\partial^{(k-1)}\Gamma_1(\Pi_n)}{\partial\Pi_n}.\end{aligned}$$

On construit un champ de directions en coupant V^n par un faisceau de 2-plans de P^{n+1}. En le prenant comme champ de référence pour un champ de Chern, on obtient les formules (2) à l'aide de (3) et d'une autre formule de l'article cité.

P. Dolbeault (Paris).

Samuel, Pierre. Singularités des variétés algébriques. Bull. Soc. Math. France **79**, 121–129 (1951).

Let V be a d-dimensional algebraic variety over an algebraically closed ground field, P a point of V, R its quotient ring, M the maximal ideal of R. Let $d+s$ denote the dimension of the vector space M/M^2 over R/M; it is well known that $s \geqq 0$, and that $s=0$ is characteristic of simple points. It is here proved that there exists a projection of V, biregular at P, onto a variety lying in a space of dimension $d+s$ if $s>0$ and of dimension $d+1$ if $s=0$. For an arbitrary local ring R of dimension d it is proved that M/M^2 has dimension d or $d+1$ over R/M if and only if there exists an ideal Q generated by a system of parameters and whose length equals its exponent.

Let C now be a plane curve $F(X, Y)=0$ over K, (x, y) being the general point. A curve $G(X, Y)=0$ is called adjoint to C at P if $G(x, y)$ is contained in the conductor Γ of the quotient ring R of P; this is shown to be equivalent to the usual condition when P is an ordinary multiple point. A differential $z dx/F_Y(x, y)$ is finite at P if and only if $z \in \Gamma$. If $\bar{R}$ is the integral closure of R, then

$$\dim_k (\bar{R}/\Gamma) = 2 \dim_k (R/K).$$

Corresponding global theorems for the coordinate ring $K[x, y]$ are also given. *I. S. Cohen* (Cambridge, Mass.).

Segre, Beniamino. Sullo scioglimento delle singolarità delle varietà algebriche. Ann. Mat. Pura Appl. (4) **33**, 5–48 (1952).

In reviewing a recent paper by Derwidué [Math. Ann. **123**, 302–330 (1951); these Rev. **13**, 67] Zariski held out very little hope indeed that it might prove possible to remove the singularities of algebraic manifolds by use of elementary (monoidal) transformations only, remarking that "the deeper difficulties of the resolution problem can be affected by such straightforward tools only to a very limited extent". The paper now to be reviewed, however, is evidence enough that hope still persists; for this paper sets out in effect to find ways and means of attempting the resolution by use only of the straightforward tools in question. The advantages aimed at, plainly, are the simplicity of conception that such a resolution would have, and its ability, by untying the singularities one-by-one and step-by-step, to give insight into their structure. In this paper, apart from much preliminary investigation, the principal result claimed is the establishment of a method for removing the singularities of surfaces by elementary transformations only. This would indeed be a notable result; but the claim, in the judgement of the reviewer, is at present unjustified, since the argument at some important points is either obscure or not as yet complete.

The paper aims to investigate the possibility that any irreducible algebraic manifold V_d (in the complex domain) can be birationally transformed by a finite sequence of elementary transformations—here called "dilatations"—into a nonsingular manifold V_d'. It contains five parts of which the first is concerned with the definition and general properties of dilatations. A dilatation, apart from certain restrictions, is the usual elementary birational transformation of a V_d in which some selected irreducible (simple or multiple) submanifold T_δ of V_d ($0 \leqq \delta \leqq d-1$) is transformed or "blown up" into a submanifold T'_{d-1} of the transform V_d' of V_d. T'_{d-1} is the model of all the tangent $[\delta+1]$ to V_d at points of T_δ; and from each point P of T_δ there arises a manifold $\phi_{d-\delta-1}$ of corresponding points P' of T'_{d-1}. The essential restrictions are, first, that T_δ must be (i) irreducible, (ii) nonsingular, and (iii) such that every one of its points has the same multiplicity for V_d (this is a key condition); and, secondly, that the transformation of V_d into V_d' must be regular (in an easily defined sense) at every point of V_d not on T_δ. This last condition, as the author proves, is satisfied if V_d' is taken to be the projective model of all sections of V_d by primals of given order l through T_δ, provided only that l exceeds the order of T_δ. In most applications in this paper, V_d is immersed in a nonsingular ambient manifold U_l ($l>d$), and the dilatation of V_d into V_d' with T_δ as base is subordinate to the dilatation of U_l into U_l' with the same base. The first fundamental property of any dilatation is that no point of T'_{d-1} can have a greater multiplicity on V_d' than the multiplicity of T_δ on V_d.

In Part II, the author, envisaging proofs by induction, first formulates his fundamental hypothesis $H(i)$: Every irreducible V_i can be transformed by a finite sequence of dilatations into a nonsingular V_i'. His proposed general plan is to assume $H(i)$ for $i<d$, and then to prove $H(d)$ by showing that, if m is the maximum multiplicity of any point of V_d, it is always possible to pass, by dilatations only, to a transform V_d' with multiplicity maximum $<m$. In the meantime, however, he wishes to obtain the following essential preliminary result: (A) If $H(i)$ ($i<d$) is assumed, and if m is defined as above, than all m-ple $(d-1)$-folds of V_d can be removed by dilatations. (A) would follow, as the author shows, if he can establish the simpler result: (A′) If T_{d-1} is irreducible, nonsingular, and everywhere of multiplicity m ($\geqq 2$) on V_d, then T_{d-1} can be replaced, by dilatations only, by a finite set of manifolds T'_{d-1} each of which is of multiplicity $<m$ on a transform V_d' of V_d. The proof of (A′), as might be expected, depends on an examination of the singularity, at a point P on T_{d-1}, of the curve in which V_d is met by a generic space of the appropriate dimension through P; but this proof appears to rely, without reservation, on too broad an interpretation of the finiteness of a set of infinitely near multiple points of an irreducible curve. This broader interpretation, already foreshadowed in the first paragraph of Part II (p. 16), arises from the suggestion that, for example, the composition of the singularity of a curve branch γ can be indifferently determined either by point dilatations of the ambient space (the usual method) or by dilatations whose bases T_δ are only required to pass through the origin of the relevant branch without touching the branch tangent. Now, for one thing, this latter mechanism (using T_δ with $\delta>0$) need not terminate; for if γ lies, for example, on a nonsingular surface ϕ of the ambient, its singularity will be totally unaffected by any dilatation (of the ambient) whose base is a curve lying on ϕ. And, for another, little appears to be known about the effect on a branch γ of such "exterior" dilatations ($\delta>0$); but it can be expected to be quite different in general from that of the normal point dilatations. These considerations, as also, for example, the question (p. 19) whether $C^{(\lambda)}$ is the complete intersection of $S^{(\lambda)}$ and $V^{(\lambda)}$, seem to leave the validity of (A′), and therefore of (A), in suspense; and they again become relevant in some proofs of Part IV.

A second important theorem of Part II tells us briefly that when the dilatation of any m-ple manifold T_δ of V_d gives rise, effectively, to an m-ple $(d-1)$-fold T^*_{d-1} of V_d', then T^*_{d-1} is always both irreducible and nonsingular, i.e., eligible as base for a dilatation of V_d'. (Here the usual T'_{d-1} is to be reckoned as T^*_{d-1} counted m-fold.)

In Part III, there is introduced a new conception, that of manifolds W_d, in an ambient nonsingular U_{d+1}, that are

"associated" to a given V_d. There is a lengthy exposition followed by a fundamental theorem. The reviewer was puzzled by some features of the exposition; but it appears that a new version of the definitions and proof of the theorem are now in course of publication, so that a brief summary of the intended role of the new development will suffice now. Being given an m-ple manifold T_δ of V_d, a possible base for a dilatation of V_d, what one needs for finiteness proofs of the kind envisaged by the author is to be able to choose at will a manifold W_d that can be relied on to contain effectively, and to an assigned multiplicity i $(1 \leqq i \leqq m-1)$, every m-ple manifold $T'_{\delta'}$, successive to T_δ, that can be made explicit by dilatations based on T_δ and its successors. If the ambient is a projective space S_{d+1}, so that V_d is a primal, the polar primals of V_d (first, second, $\cdots$ polars) make some attempt to fulfil the role in question, but not reliably, as is now known. The intention is, then, to define a more general class of primals "associated" to V_d, from among which we shall be able to choose some that will do what we want. To do this (still in S_{d+1}) we can say that a primal W_d has "associated behavior (of index 1)" with V_d at P if the equations $f(x)=0$ and $g(x)=0$ of V_d and W_d (in nonhomogeneous coordinates $x_1, \cdots, x_{d+1}$) are such that there exist polynomials $a(x), b(x), b_1(x), \cdots, b_{d+1}(x)$ with the properties (i) $a(x)$ does not vanish at P, and (ii) $a(x)g(x)=b(x)f(x)+\sum_1^{d+1} b_i(x)\partial f/\partial x_i$ identically in the x_i. If V_d has an m-ple point at P, then W_d has at least an $(m-1)$-ple point at P; and the association is said to be "regular" if W_d has precisely an $(m-1)$-ple point at P. Similarly, by an inductive procedure, we define "associated behavior, and regular associated behavior, of any index i" at P $(1 \leqq i \leqq m-1)$, the corresponding minimum or exact multiplicity of W_d at P being $m-i$. The essential extension of these concepts to "primals" on any nonsingular ambient U_{d+1} would seem to need expression in terms of power series; and it appears that, in the author's new paper, the appropriate definitions can be formulated and shown to be independent of the choice of local coordinates on U_{d+1}. The fundamental theorem of this part may be summarized as follows: (B) If V_d is an irreducible "primal" (divisor) in a nonsingular ambient manifold U_{d+1}, and if a submanifold T_δ, m-ple for V_d, is dilated into $T'_{d-1} \subset V_d' \subset U'_{d+1}$, then any primal W_d of U_{d+1} that has regular associated behavior of index i with V_d at a point P of T_δ is transformed into a primal W_d' of U'_{d+1} that has associated behavior of index i with V_d' at every point P' which arises from P; and, further, if P' is again m-ple for V_d', the associated behavior at P' is again regular. In other words, initial regularity of the behavior is to ensure that W_d actually passes (to multiplicity $m-i$) through every consecutive m-ple manifold T_δ', of V_d that can be explicitly exhibited by dilatations. Whatever the method of defining associated manifolds W_d, it would seem to be necessary, for the applications in view, that we should always be able to find some among them that are algebraic.

In Part IV the author aims to establish the reduction hypothesis for surfaces. He supposes V_3 to be immersed in an ambient U_3, and proceeds as follows. (i) By appeal to (A), V_3 is first supposed to be freed, by dilatations, from proper m-ple curves, where m is the multiplicity maximum on the surface. (ii) Everything now reduces to proving the following: (C) If m is the multiplicity maximum on V_3, every simple consecutive sequence of isolated m-ple points and m-ple curves (arbitrarily interspersed) that starts from a proper isolated m-ple point of V_3 and is progressively defined by dilatations is necessarily finite. (Each dilatation base, after the first, must be an isolated m-ple point or an m-ple curve arising from the preceding one, and the m-ple point may be one of several that are available.) (iii) The proof of (C) for arbitrary m is reduced to that for the case $m=2$, i.e., for the case of a surface which has only a finite number of isolated double points. This reduction, based on the assumption of a "generic" surface W_3 (necessarily algebraic) which has (regular) associated behavior of index $m-2$ with V_3 at an (isolated) m-ple point of the latter, was by no means clear to the reviewer, especially as regards the last two paragraphs of the proof (pp. 38, 39). (iv) The proof of (C), finally, for a surface V_3 with only isolated double points, depends on the use of a "generic" surface W_3 that has regular associated behavior with V_3 at a double point P of the latter, the idea being to show that at least one component of the curve $V_3 \cdot W_3$ must pass through all the points of the mixed sequence referred to in (C), and that this would be impossible if the sequence in question were infinite. To establish the existence of such a component, the author considers the intersection curve $V_3' \cdot W_3'$ of the transforms of V_3, W_3 at one of the points in question; and his main task is to show that not every branch of this curve at the point arises from an exceptional curve of W_3' generated in the course of the dilatation process. This involves some detailed analysis of properties of dilatations which have various types of double point and double curve of a surface as bases. Once this difficulty is surmounted, the plan of the proof is complete.

In Part V, finally, the problem for $d \geqq 3$ is surveyed. The principal new difficulty that arises is that the preliminary removal of the singularities of an m-ple manifold T_δ of V_d (to prepare it for reduction by dilatation) may give rise to new m-ple manifolds which may themselves be singular; and again the dilatation of a nonsingular m-ple T_δ may produce similar unwelcome effects. The author confines himself here to some general indications of possible lines of procedure.

Of the investigation as a whole one can say that it certainly outlines a bold plan; but the possibility of proof—even for surfaces—is still very much in suspense.

J. G. Semple (London).

Differential Geometry

Strubecker, Karl. Kinematik, Liesche Kreisgeometrie und Geraden-Kugel-Transformation. Elemente der Math. **8**, 4–13 (1953).

This is an expository lecture on Study's [Monatsh. Math. Phys. **1**, 283–355 (1890); Math. Ann. **39**, 441–566 (1891)] treatment of rigid motions in a plane by means of modified quaternions, and the induced representation of the motions by the points of a 3-space. The lecture includes a discussion of more recent work, chiefly by Blaschke [Z. Math. Physik **60**, 61–91, 203–204 (1911)], Grünwald [S.-B. Akad. Wiss. Wien. Math.-Nat. Kl. IIa. **120**, 677–741 (1911)], and the author [ibid. **139**, 685–705 (1930); Jber. Deutsch. Math.-Verein. **44**, 184–198 (1934)], which relates Study's conceptions to the Lie circle geometry and the theory of line-sphere transformations. *L. A. MacColl* (New York, N. Y.).

Hohenberg, Fritz. Komplexe Erweiterung der gewöhnlichen Schraublinie. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **160**, 15–29 (1951).

If the coordinates of a point $P(x, y, z)$ on the helix $x=r\cos\phi$, $y=r\sin\phi$, $z=p\phi$ assume the complex values

$z=x_1+ix_2$, $y=y_1+iy_2$, $z=z_1+iz_2$, the complex point is representable by a pair of real points $P_1(x_1, y_1, z_1)$ and $P'(x', y', z')$ where $x'=x_1+x_2$, $y'=y_1+y_2$, $z'=z_1+z_2$. If $\phi=\phi_1+i\phi_2$, then

P_1: $x_1=r\cos\phi_1\cosh\phi_2$, $y_1=r\sin\phi_1\cosh\phi_2$, $z_1=p\phi_1$

P': $x'=r(\cos\phi_1\cosh\phi_2-\sin\phi_1\sinh\phi_2)$,

$y'=r(\sin\phi_1\cosh\phi_2+\cos\phi_1\sinh\phi_2)$, $z'=p(\phi_1+\phi_2)$.

The helicoidal surfaces generated by P_1 and P' are also translation surfaces. Furthermore, the P'-surface may be generated by moving a helix with a screw motion about a line parallel to its axis. Many geometrical properties of the surfaces are derived. For example, if $\phi_1=0$, the parametric line traced on the P'-surface as ϕ_2 is varied is a skew curve whose elevation is a parabola and whose plan is a pseudocycloid. The fundamental forms, curvatures, and correspondences of the surfaces are derived. *M. Goldberg.*

Green, J. W. Curves encircling a cylinder. Amer. Math. Monthly **60**, 30–31 (1953).

The author considers curves of diameter 2 encircling a right circular cylinder of unit radius and proves that such a curve is always rectifiable and its length is $\leqq 2^{3/2}\pi$. Furthermore it always can be included between two parallel planes of distance $\leqq 2^{1/2}$ perpendicular to the cylinder. These constants cannot be replaced by smaller ones.

L. Fejes Tóth (Veszprém).

Togliatti, Eugenio. Sulla geometria intrinseca di un gruppo continuo di trasformazioni. Rend. Sem. Mat. Fis. Milano **22** (1951), 90–102 (1952).

A construction of differential invariants of plane curves under a continuous group of transformations with the method of Pick and Kowalewski. *A. Nijenhuis.*

Finzi, Arrigo. Sur le problème de la génération d'une transformation donnée d'une courbe fermée par une transformation infinitésimale. Ann. Sci. Ecole Norm. Sup. (3) **69**, 371–430 (1952).

Previously [same Ann. (3) **67**, 243–305 (1950); these Rev. **12**, 434] the author has given a sufficient condition that a transformation $T: y=g(x)$ upon a closed curve be generated by an infinitesimal transformation $\xi(x)$. The present paper gives another criterion based on the following theorem: if $T(\theta): y=g(x, \theta)$ is a family of transformations regular in the sense of the former paper, if g_x, g_θ, g_{xx}, $g_{x\theta}$ satisfy a Lipschitz condition with respect to x, θ, and if $0<g_\theta$, there is a family $\xi(x, \theta)$ of infinitesimal transformations with groups having elements $T(\theta, t)$ and $T(\theta, 1)=T(\theta)$.

J. M. Thomas (Durham, N. C.).

Schönhardt, Erich. Über die Projektion von Vektoren auf Kurven und eine gewisse Kurventransformation. Arch. Math. **3**, 314–326 (1952).

Let $\mathfrak{t}=\mathfrak{t}(s)$ be the tangent vector of a plane curve $\mathfrak{C}$ [$\mathfrak{t}^2=1$; $0\leqq s\leqq l$]. The "projection" of a plane vector $\mathfrak{V}$ is the vector $\mathfrak{V}(\mathfrak{C})=l^{-1}\int_0^l(\mathfrak{V}\mathfrak{t})\mathfrak{t}ds$. The linear transformation $\Phi: \mathfrak{V}\to\mathfrak{V}(\mathfrak{C})$ has the trace 1 and $0\leqq\det\Phi\leqq 1/4$. The effect on Φ of rotations and reflections of $\mathfrak{C}$ is determined. Furthermore, the curves $\mathfrak{C}$ are discussed for which $\det\Phi=1/4$. For them 2Φ is the identity. *P. Scherk.*

Muracchini, Luigi. Contributo alla geometria proiettiva differenziale dei 3-tessuti di curve piane. Boll. Un. Mat. Ital. (3) **7**, 285–292 (1952).

There are obtained a complete system of projective invariants for a general 3-web of curves in a projective plane and a system of Pfaffian equations for studying the problem of projective deformation of two 3-webs of plane curves. The tangential projectivity, associated to a tangent at a point of a curve of a 3-web and introduced by Bompiani [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 7–12 (1949); these Rev. **10**, 625], and the conditions for two 3-webs of plane curves to be projectively deformable to each other are expressed in terms of the aforesaid projective invariants of the 3-webs. *C. C. Hsiung.*

Negri, Domenico. Sopra un determinante studiato dal prof. Ascoli. Atti Sem. Mat. Fis. Univ. Modena **5**, 101–104 (1951).

An expansion is given for the determinant

$$\begin{vmatrix} T_u\times P_u & T_u\times P_v & T_u\times n \\ T_v\times P_u & T_v\times P_v & T_v\times n \\ T\times P_u & T\times P_v & T\times n \end{vmatrix},$$

where n is the unit normal to a surface with position vector P and parameters (u, v) in Euclidean 3-space, and T is a unit vector. The determinant was introduced by G. Ascoli [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 280–285 (1948); these Rev. **10**, 570]. *J. M. Thomas.*

Ströher, Wolfgang. Darstellung des Linienelementes sechster Ordnung durch W-Kurven. Monatsh. Math. **56**, 288–303 (1952).

A W-curve depends on 8 parameters and consequently it is defined by its line element of order seven at a given point P. Hence all W-curves with a common line element of order 6 (at P) constitute a one-parametric set S. The author's chief scope is the investigation of this set. A very useful tool for this investigation is the Kowalewski relative coordinates subject to the "identity conditions" (which are the projective generalization of the Cesàro "fixed point conditions"). They lead to a cubic curve C^3 of inscribed fundamental triangles of the W-curves with constant projective curvatures. The author investigates thoroughly the corresponding 1-2 involution of the vertices, in particular, with respect to double elements. Let w be a particular W-curve of S, C^3 its osculating conic section in P, and K^3 a tangent conic section which has the fundamental triangle of w for its polar triangle. Then the Mehmke invariant of C^3 and K^3 is $M=-1$. The projective normal of W at P is the common chord of C^3 and K^3, which are autopolar.

V. Hlavatý (Bloomington, Ind.).

Flora, Miltiades. On the congruences of Guichard. Bull. Soc. Math. Grèce **26**, 29–68 (1952). (Greek)

A Guichard congruence is a line congruence in E^3 where the developable surfaces of the congruence intersect the focal surfaces in lines of curvature. The principal result is: The two focal surfaces of a Guichard congruence are Weingarten surfaces if and only if under the mapping of one focal surface on the other which associates the intersection with the same line of the congruence the ratio of the areas of corresponding sets is constant. *H. Busemann.*

Semin, Ferruh. On Darboux lines. Rev. Fac. Sci. Univ. Istanbul (A) **17**, 351–383 (1952). (Turkish summary)

Les lignes de Darboux (lignes D) d'une surface S sont les courbes le long desquelles la sphère osculatrice est tangente à S. Elles sont caractérisées par la relation $\rho_g\cdot\tau_g+d\rho_n/ds=0$, où ρ_g, τ_g, ρ_n désignent respectivement la courbure géodésique, la torsion géodesique, et la courbure normale (s étant l'arc), et leur détermination dépend d'une

certaine équation différentielle du deuxième ordre. L'auteur montre que pour une courbe distincte d'une circonférence, deux des trois conditions suivantes: la courbe est D, ou bien courbe de Monge, ou bien géodésique de S, entraînent la troisième. Il considère ensuite les lignes D tangentes en un point à une ligne de courbure ou une asymptotique de S. Les plans osculateurs des courbes D au point envisagé sont en général tangents à S. Dans le cas du contact avec une ligne de courbure le rayon de courbure de la ligne D au point de contact est nul; dans le cas du contact avec une ligne asymptotique la torsion de la ligne D au même point est nulle, et sa courbure vaut les 2/3 de celle de l'asymptotique tangente. Entre une géodésique et une ligne D tangentes en un point, il existe la relation $\overline{P_nP_0^g}+3\overline{P_nP'}=0$, ou P_n et P_0^g sont respectivement le centre de courbure et le centre de la sphère osculatrice à la géodésique au point envisagé, et où P' est le point où la droite $P_nP_0^g$ coupe la normale principale de la ligne D en ce même point. Le lieu des centres de courbure des lignes D d'une surface en un point P de celle-ci est une courbe algébrique unicursale, en général du 8ème degré, et le fait que cette courbe est plane caractérise les quadriques véritables de révolution, les cylindres généraux et les cônes de révolution. L'auteur recherche, en terminant, les lignes D d'une surface de révolution, et il montre que leur détermination se ramène à des quadratures. *P. Vincensini.*

Terracini, Alessandro. Il caso singolare nella determinazione di una superficie di S_5 a partire dalle sue linee principali. Univ. e Politecnico Torino. Rend. Sem. Mat. **11**, 359–373 (1952).

L'auteur revient sur le problème de l'existence d'une surface de l'espace à cinq dimensions admettant des lignes principales assignées, existence demontrée par lui dans un mémoire antérieur [Union Mat. Argentina, Publ. no. 16 (1940); ces Rev. **2**, 300]. Dans le mémoire indiqué, le cas ou les cinq systèmes de lignes principales ne sont pas tous confondus est traité complètement, mais la recherche de toutes les surfaces admettant un système quintuple de lignes principales restait à faire, l'auteur s'étant borné à signaler l'existence de telles surfaces, existence résultant de celle des surfaces admettant un système quintuple de lignes principales planes déterminées par lui dans un autre travail [Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **73**, 317–336 (1938)]. Il traite ici complètement le cas où la circonstance singulière de la coincidence des cinq familles de lignes principales se présente, et montre que les seules surfaces donnant lieu à la particularité indiquée sont les surfaces admettant un système quintuple de lignes principales planes dont il vient d'être question, et deux autres types très particuliers, ne dépendant que de constantes arbitraires, et pour l'un desquels les surfaces appartiennent à la classe des surfaces W admettant un groupe de ∞^2 homographies transformant les surfaces en elles- mêmes. L'étude de l'ordre d'approximation de l'incidence des plans infiniment voisins le long des lignes principales montre que cet ordre, qui est ≥ 4 pour une surface quelconque de l'espace, est ∞ pour les surfaces à lignes principales planes, et est égal à 8 ou 10 pour les surfaces de l'un et de l'autre des deux autres types speciaux. *P. Vincensini* (Marseille).

Morozova, E. A. Shortest lines on surfaces of revolution with a rectifiable meridian. Doklady Akad. Nauk SSSR (N.S.) **84**, 1135–1138 (1952). (Russian)

Let the surface F be generated by revolving a rectifiable curve C about a line L not intersecting C. As parameters on F take the arc length s on C and angle v of revolution. Any shortest connection G of two points on F which is not a meridian may be represented in the form $s=s(v)$ and satisfies the differential equation $ds/dv=\pm r(s)(r^2(s)-k^2)^{1/2}/k$, where k is a constant depending on G, and $r(s)$ is the distance of (s, v) from L. At a point p of G where the meridian M through p has right and left tangents the curve G does, too. If $\alpha^r(v)$ and $\alpha^l(v)$ denote the angles of these tangents with M, then $r(s)\sin\alpha^r(v)=r(s)\sin\alpha^l(v)=k$, provided the ratio of the length of the chord from p to $q\in M$ to the length of the arc pq of M tends to 1 for $q\to p$. *H. Busemann.*

Ritter, Robert. Über gewisse Zwischenintegrale der Biegungsgleichung spezieller Flächen. Arch. Math. **3**, 395–400 (1952).

The construction of a surface by means of its fundamental tensor $a_{\lambda\mu}$ is based on the Monge-Ampère equation

$$(1)\qquad (1-\Delta_1 x)K=A^{-2}(x_{I\,I}\,x_{II\,II}-x^2{}_{I\,II})$$

[cf., for instance, Hlavatý, Differentialgeometrie der Kurven und Flächen und Tensorrechnung, Noordhoff, Groningen-Batavia, 1939, p. 430; these Rev. **1**, 27] where K is the Gauss curvature, Δ_1 is the first differential operator, A^2 the discriminant, and $x_{\lambda\mu}$ the second covariant derivative. If we put $\rho=K^{-1/2}\neq 0$, $\theta(a, b)\equiv A^{-1}(a_I\,b_{II}-a_{II}\,b_I)$ and assume

$$\rho\Delta_2\rho=-\tfrac{1}{2}\Delta_1\rho+4,\quad \rho\Delta(\rho\Delta_1\rho)=-\Delta_1\rho(\Delta_1\rho-4),\quad \rho\theta(\rho_1\Delta_1\rho)=0,$$

then (1) admits the intermediate integrals

$$\rho^{1/2}[\tfrac{1}{2}\theta(\rho, x)\pm[-(1-\Delta_1(x)]^{1/2}].$$

The proof follows by straightforward covariant computation. *V. Hlavatý* (Bloomington, Ind.).

Schindler, Johann Jakob. Beiträge zur Theorie von Stützfunktion und Radius. Comment. Math. Helv. **26**, 242–251 (1952).

Let R denote the square of the length of the radius vector to a point of a surface $x=x(u, v)$, and let P be the supporting function of the surface at this point. The author in this paper determines the surfaces for which there is a functional relation between R and P. A construction is given for such surfaces by identifying certain elements on the surface with the curvature, torsion, and moving trihedral of an arbitrary space curve. The surfaces of revolution are the surfaces obtained by this process from the helices.

S. B. Jackson (College Park, Md.).

Berri, R. Ya. On an integer invariant of binary forms of the fourth order. Uspehi Matem. Nauk (N.S.) **7**, no. 3(49), 125–130 (1952). (Russian)

Let $F(x, y)$ be a given form of degree m and $\phi(x, y)$ an arbitrary form of degree $n>m$. If (*) $F_{xx}\phi_{yy}-2F_{xy}\phi_{xy}+F_{yy}\phi_{xx}\equiv 0$ implies $\phi\equiv 0$ for $n=m+1, \cdots, m+h$, but not for $n>m+h$, we put $N(F)=h$. If there is a $\phi\not\equiv 0$ of degree $m+1$ satisfying (*), we put $N(F)=0$. The number $N(F)$ plays an important part in the investigations of isometric deformation of a surface in the neighborhood of a flat point. For $m=4$ (the first nontrivial case), it is shown that $N(F)\leq 1$, and $N(F)=1$ except for certain special cases. *H. Busemann.*

Efimov, N. V. Some propositions on rigidity and nondeformability. Uspehi Matem. Nauk (N.S.) **7**, no. 5(51), 215–224 (1952). (Russian)

If an analytic surface S vectorially represented by $\mathfrak{x}(u, v)$ is infinitesimally rigid of order 1, then it is obviously rigid, i.e., cannot be isometrically deformed in an analytic way except by motions. A simple proof is given for the fact that

infinitesimal rigidity of order 2 also implies rigidity (a more complicated proof is found in an earlier paper of the author [Uspehi Matem. Nauk (N.S.) **3**, no. 2(24), 47–158 (1948); these Rev. **10**, 324]). Whether a similar statement is true for infinitesimal rigidity of higher order is an open question. The following is proved for order 3: If the surface is infinitesimally rigid of order 3 and if there are no two independent nonconstant vector fields $\mathfrak{y}_1$, $\mathfrak{y}_2$ on the surface for which $[\mathfrak{y}_1\, d\mathfrak{x}]$ and $[\mathfrak{y}_2\, d\mathfrak{x}]$ are both total differentials, then S is rigid. *H. Busemann* (Los Angeles, Calif.).

Tartakovskiĭ, V. A. On N. V. Efimov's N-invariants from the theory of deformation of surfaces. Mat. Sbornik N.S. **32**(74), 225–248 (1953). (Russian)

Let $F=\sum_{0\le p,q\le n}^{p+q=n}\lambda_{pq}x^p y^q$ be a form of degree n with real coefficients. If no form ϕ of degree greater than n exists for which $F_{xx}\phi_{yy}-2F_{xy}\phi_{yx}+F_{yy}\phi_{xx}\equiv 0$, then F is called normal, otherwise special. We associate with F the point $(\lambda_0, \cdots, \lambda_n)$ in the $(n+1)$-dimensional space Λ_n. If Λ_n' denotes the subset of Λ_n corresponding to special forms, then $\Lambda_n'=\Lambda_n$ for $n=1, 2, 3, 4$. For $n\ge 5$, the set Λ_n' is the union of a countable number of algebraic varieties of dimension less than $n+1$. Obviously the set Λ_n^0 corresponding to all forms F which are either special or contain a multiple linear factor is still the union of a countable set of algebraic manifolds of dimension less than $n+1$, so that $\Lambda_n-\Lambda_n^0$ is not empty. Explicit examples of functions in $\Lambda_n-\Lambda_n^0$ are constructed for each $n\ge 5$. This result is of considerable importance for the theory of deformation of a surface in the neighborhood of a flat point [see Efimov, Trudy Mat. Inst. Steklov. **30** (1949); these Rev. **12**, 531]. *H. Busemann.*

Ancochea, Germán. Affine und projektive Differentialgeometrie der singulären ebenen Kurvenelemente. Abh. Math. Sem. Univ. Hamburg **18**, 1–13 (1952).

Anschliessend an Arbeiten von Bompiani [Univ. Roma Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **5**, 1–46 (1946); diese Rev. **9**, 60] und Maxia [ibid. **6**, 253–345 (1947); diese Rev. **9**, 530] gibt Verf. eine Theorie der singulären ebenen Kurvenelemente ohne irgendeine Einschränkung für Ordnung und Klasse und er bestimmt für jeden Fall ein vollständiges Invariantensystem. Er fängt an in der affinen Geometrie und zeigt dass für einen analytischen Zweig der Ordnung m und der Klasse $n-m$ immer eine Parameterdarstellung

$$x=t^m+t^{m+r}+\lambda_{r+1}t^{m+r+1}+\cdots, \quad y=t^n$$

gefunden werden kann, wobei $\lambda_{n-m}=0$ ist. Dabei ist die X-Achse längs der Tangente, die Y-Achse längs der Affinnormale gewählt worden und t in geeigneter Weise normiert. Die Koeffizienten λ dieser kanonischen Darstellung bilden ein vollständiges System von absoluten Invarianten. Die projektive Differentialgeometrie bringt Verf. auf die affine zurück indem es ihm gelingt eine Gerade zu bestimmen, die auf innere projektive Weise mit dem Zweig verbunden ist und die er dann als uneigentliche Gerade benutzen kann. Die "Affinnormale" bezüglich dieser Gerade ist die Projektivnormale. Bemerkt sei noch dass die Affininvarianten der Differentialordnungen $<n+m$ gleichzeitig Projektivinvarianten sind. *O. Bottema* (Delft).

Barner, Martin. Zur projektiven Differentialgeometrie der Kurvenpaare. Math. Z. **56**, 409–442 (1952).

The author studies the projective differential geometry, in a three-dimensional projective space, of a pair of curves C, $\bar{C}$ at two points $R(t)$, $\bar{R}(t)$ corresponding to a same parameter t under the conditions that the osculating plane of the curve C ($\bar{C}$) at the point R ($\bar{R}$) does not contain the point $\bar{R}$ (R) and the tangents of the curves C, $\bar{C}$ respectively at the points R, $\bar{R}$ do not meet together. Let S ($\bar{S}$) be the point of intersection of the tangent to the curve C ($\bar{C}$) at the point R ($\bar{R}$) with the osculating plane of the curve $\bar{C}$ (C) at the point $\bar{R}$ (R), and let $\mathfrak{r}(t)$, $\mathfrak{s}(t)$, $\bar{\mathfrak{s}}(t)$, $\bar{\mathfrak{r}}(t)$ be respectively the vector coordinates of the points R, S, $\bar{S}$, $\bar{R}$ with respect to a projective coordinate system; then the following Frenet equations are obtained:

$$\mathfrak{r}'=a\mathfrak{s}, \quad \mathfrak{s}'=\bar{A}\,\mathfrak{r}+b\,\bar{\mathfrak{s}},$$
$$\bar{\mathfrak{r}}'=\bar{a}\,\bar{\mathfrak{s}}, \quad \bar{\mathfrak{s}}'=A\,\bar{\mathfrak{r}}+\bar{b}\,\mathfrak{s},$$

where the prime denotes differentiation with respect to t. The quantities

$$(*) \qquad \frac{b\,\bar{b}}{a\,\bar{A}}, \quad \frac{b\,\bar{b}}{\bar{a}\,A}, \quad \frac{a\,\bar{A}}{\bar{a}\,A}$$

are invariant under transformations of the parameter t as well as under those of proportionality factors of $\mathfrak{r}$, $\mathfrak{s}$, $\bar{\mathfrak{s}}$, $\bar{\mathfrak{r}}$. Two pairs of curves can be in a projective correspondence such that corresponding points have a same parametric value if and only if the two pairs of curves have the same functions

$$a\,\bar{A}, \quad \bar{a}\,A, \quad b\,\bar{b}, \quad \left(\ln\frac{b}{a}\right)', \quad \left(\ln\frac{\bar{b}}{\bar{a}}\right)', \quad \left(\ln\frac{a}{\bar{a}}\right)'.$$

Using the points R, S, $\bar{S}$, $\bar{R}$ as the vertices of a local tetrahedron of reference, the local coordinates of a point on the curve C in the neighborhood of the point R are expressed in power series of t. Local plane coordinates of a plane are also defined and the incidence condition for a point with a plane is obtained.

In the second section, correlations, projectivities and various osculating cubic curves, cones, conics and quadrics associated to the pair of curves are discussed, and by using these osculating configurations the invariants in (*) are interpreted in terms of certain cross ratios.

In the last section, the osculating configurations defined in the second section are further studied in connection with the ruled surfaces respectively generated by the lines $(\mathfrak{r}, \bar{\mathfrak{r}})$, $(\mathfrak{s}, \bar{\mathfrak{s}})$, $(\mathfrak{r}, \bar{\mathfrak{s}})$, $(\bar{\mathfrak{r}}, \mathfrak{s})$ as the parameter t varies, so that the invariants

$$\frac{(\ln \bar{a}/a)'}{2(-b\,\bar{b})^{1/2}}, \quad -\frac{(\ln b/a)'}{3(-b\,\bar{b})^{1/2}}, \quad \frac{(\ln \bar{b}/\bar{a})'}{3(-b\,\bar{b})^{1/2}}$$

can be expressed in terms of cross ratios; and thus a complete invariant system of the pair of curves is geometrically interpreted. *C. C. Hsiung* (Bethlehem, Pa.).

Gol'dšteĭn, L. V. Construction of a projective theory of curves by means of central affine geometry. Trudy Sem. Vektor. Tenzor. Analizu **9**, 288–308 (1952). (Russian)

In the present paper the curve theory in a two- or three-dimensional projective space is reconstructed, i.e., the projective length $\sigma=\int p^{1/3}du$ defined by the integral invariant p of the lowest order and the projective curvatures of a curve are calculated from the standpoint of central-affine geometry, as usual, making use of absolute differentiation in a one-dimensional space and of projective normalization of point coordinates. It is also verified that $p=0$ characterizes a conic in a projective plane or a curve belonging to a linear complex in a projective space; and then the Frenet formulas are obtained. The method used is not different from the usual one and the reviewer could not find original

concepts or new results. The author seems to be not familiar with the book by G. Fubini and E. Čech [Geometria proiettiva differenziale, t. I, Zanichelli, Bologna, 1926].

A. Kawaguchi (Sapporo).

Wong, Yung-Chow. A note on Levine's paper: "Fields of parallel vectors in projectively flat spaces". Duke Math. J. **20**, 119–125 (1953).

Let $B^i{}_{jkl}$ be the curvature tensor of an affine space A_n. Put $B_{jk}=B^i{}_{jki}$ and $P_{jk}=(n^2-1)^{-1}(nB_{jk}+B_{kj})$. Assume that A_n is projectively flat. Then a necessary condition that it admit a field of contravariant (covariant) parallel vectors λ^i (μ_j) is

$$\lambda^i P_{[kl]}=\lambda^j P_{j[k}\delta^i_{l]} \quad (P_{[kl]}\mu_j=P_{j[k}\mu_{l]}).$$

It is easily seen that this condition is equivalent to

$$P_{[jk]}=0 \text{ for } n>2 \quad (2P_{[jk]}+P_{jk}=\nu_j\mu_k).$$

These conditions serve as a basis for a set of theorems closely connected with the Levine results [J. Levine, same J. **16**, 23–32 (1949); these Rev. **10**, 480]. The case $n=2$ for contravariant fields is carried through in the last section and the corresponding $\Gamma^i{}_{jk}$ as well as the field λ are found.

V. Hlavatý (Bloomington, Ind.).

Norden, A. P. The conformal Euclidean space of Weyl. Doklady Akad. Nauk SSSR (N.S.) **50**, 53–55 (1945). (Russian)

A normalized W_n is a conformal Möbius space M_n with a correspondence between pairs of points. The geometry of such spaces has been investigated by Norden and his colleagues during the past seven or eight years [Mat. Sbornik N.S. **20**(62), 263–281 (1947); Doklady Akad. Nauk SSSR (N.S.) **61**, 207–210 (1948); these Rev. **9**, 67; **10**, 67]. The present paper (note the date) deals with the intrinsic affine connection that may be introduced in such a space and with proving that every conformal euclidean Weyl space may be interpreted as a W_n. *M. S. Knebelman.*

Chatterjee, B. C. On some geometrical configurations. I. Bull. Calcutta Math. Soc. **43**, 135–138 (1951).

The author obtains a set of ten conditions, any one of which is sufficient for the existence of a unique null element in the system defined by R. N. Sen [same Bull. **42**, 1–13 (1950); these Rev. **12**, 205]. He then defines a set of transformations which is isomorphic to Sen's system and leads to a simple geometric configuration in which the center of symmetry corresponds to the null element. *A. Fialkow.*

Mishra, R. S. Hyper-asymptotic curves of a Riemannian hypersurface. Math. Student **20**, 63–65 (1952).

In previous papers [Bull. Calcutta Math. Soc. **42**, 118–122, 213–216 (1950); these Rev. **12**, 532; **13**, 278], the author defined a hyper-asymptotic curve on a surface in Euclidean 3-space. It is a curve on a surface such that its rectifying plane at all points contains a line of a specified rectilinear congruence through that point. The present paper generalizes this definition to apply to the case of a hypersurface V_n of a Riemann space. The differential equations of the hyper-asymptotic curves are obtained. It is shown that if the curves of the defining congruence are normal to V_n, then the hyper-asymptotic lines reduce to the asymptotic lines of V_n. The hyper-asymptotic lines of V_n are characterized by the property that the resolved parts of the curvature vectors of these lines along the curve of the congruence vanish. All the proofs are straightforward. *A. Fialkow.*

Botella Raduán, F. On the reviews of three notes. Some questions of geometry in a Riemannian space. Revista Mat. Hisp.-Amer. (4) **12**, 229–233 (1952). (Spanish)

The author defends three earlier notes against criticism expressed in these Reviews and the Zentralblatt für Mathematic [same Revista (4) **1**, 163–170 (1941); **3**, 302–309 (1943); **4**, 10–15 (1944); these Rev. **7**, 34; **6**, 216; **7**, 34; Zbl. Math. **26**, 86 (1942)].

Yano, Kentaro. On n-dimensional Riemannian spaces admitting a group of motions of order $n(n-1)/2+1$. Trans. Amer. Math. Soc. **74**, 260–279 (1953).

The order $\frac{1}{2}n(n-1)+1$ for a group of motions of a V_n is of particular interest, because it is the maximum order of the complete group of motions of a V_n that is not an Einstein space. Besides, if $n\neq 4$, there are no groups of motions of order r such that $\frac{1}{2}n(n-1)+1<r<\frac{1}{2}n(n+1)$. The author's main result is (Theorem 9): A necessary and sufficient condition that an n-dimensional Riemannian space V_n for $n>4$, $n\neq 8$, admit a group G_r of motions of order $r=\frac{1}{2}n(n-1)+1$ is that the space be the product space of a straight line and an $(n-1)$-dimensional Riemannian space of constant curvature (this is equivalent to the fact that the space is conformally flat and admits a parallel vector field) or that the space be of negative constant curvature.

A. Nijenhuis (Princeton, N. J.).

Yano, Kentaro, and Adati, Tyuzi. On certain spaces admitting concircular transformations. Proc. Japan Acad. **25**, 188–195 (1949).

A concircular transformation of a Riemann space V_n is a conformal transformation of the space, $\bar{g}_{\alpha\beta}=\sigma^{-2}g_{\alpha\beta}$, which maps each geodesic circle upon a geodesic circle. The authors study Riemann spaces V_n which admit concircular transformations and are subject to various additional conditions. They begin with the differential equations which are the necessary and sufficient conditions that a conformal transformation be concircular and prove several theorems of which the following are typical: (1) A conformally flat Riemann space V_n ($n>3$) which admits a concircular transformation is a subprojective space of Kagan. Any subprojective space of Kagan admits a concircular transformation. (2) In a conformally flat V_n ($n>3$) admitting a concircular transformation, the hypersurfaces $\sigma=$constant are all spaces of constant curvature. The conditions that all these hypersurfaces be Einstein spaces are also given. *A. Fialkow.*

Matsumoto, Makoto. Some applications of Bochner's method to Riemannian manifolds. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **27**, 167–174 (1952).

The author gives sufficient conditions for the non-imbeddability of various compact, orientable, Riemannian n-manifolds V_n into euclidean $(n+1)$-space E_{n+1} and for the non-existence of certain harmonic tensors on such manifolds. The conditions are in terms of rather complicated tensors. One of the author's results states that under certain conditions V_n cannot be imbedded as a minimal variety in E_{n+1}; but these extra conditions are redundant, as the reviewer has proved [Trans. Amer. Math. Soc. **71**, 211–217 (1951); these Rev. **13**, 492] that a compact minimal variety V_n never exists in E_{n+1}. *S. B. Myers.*

Liber, A. E. On two-dimensional spaces with an algebraic metric. Trudy Sem. Vektor. Tenzor. Analizu **9**, 319–350 (1952). (Russian)

Let $F_2^{(p)}$ be a two-dimensional space endowed with a differential form of order p ($\geqq 3$): $ds^p=a_{\alpha_1\cdots\alpha_p}d\xi^{\alpha_1}\cdots d\xi^{\alpha_p}$

(a's run over 1, 2) whose discriminant $\mathfrak{A}\neq 0$. The main result of the present paper is that in $F_2^{(p)}$ one can define the invariant linear affine connection $\Gamma^{\alpha}{}_{\beta\gamma}(\xi)$ depending upon only a point and no line element. §1 is devoted to discussion of algebraic comitants, i.e., many tensor densities as well as the relations among them are derived from the fundamental tensor $a_{\alpha_1\cdots\alpha_p}$ in a two-dimensional centro-affine space. The important ones of them are the following:

$$\mathfrak{H}_{\alpha_1\cdots\alpha_{2p-4}}$$

(coefficients of the Hessian of the form $a=a_{\alpha_1\cdots\alpha_p}x^{\alpha_1}\cdots x^{\alpha_p}$), $\mathfrak{B}^{\alpha_1\cdots\alpha_{2p-4}}$ (solutions of the system of $2p-3$ independent linear equations:

$$a_{\alpha_1\cdots\alpha_p\beta}\times\mathfrak{B}^{\alpha_1\cdots\alpha_p\gamma_1\cdots\gamma_{p-3}}=0$$

and

$$\mathfrak{H}_{\alpha_1\cdots\alpha_{2p-4}}\mathfrak{B}^{\alpha_1\cdots\alpha_{2p-4}}=\mathfrak{A}),$$

$$\mathfrak{C}_{\alpha_1\cdots\alpha_p}=\mathfrak{B}^{\lambda_1\cdots\lambda_{p-2}}{}_{\alpha_1\cdots\alpha_{p-2}}\times a_{\alpha_{p-1}\alpha_p\lambda_1\cdots\lambda_{p-2}},\qquad \mathfrak{L}_{\alpha_1\cdots\alpha_p;\,\lambda_1\cdots\lambda_{p-2}}$$

(solutions of the system of $p(p-2)$ independent equations:

$$\mathfrak{B}^{\alpha_1\cdots\alpha_p\gamma_1\cdots\gamma_{p-3}}\mathfrak{L}_{\alpha_1\cdots\alpha_p;\,\lambda_1\cdots\lambda_{p-2}}=\mathfrak{A}\delta^{\gamma_1}_{\alpha_1}\cdots\delta^{\gamma_{p-3}}_{\lambda_{p-2}})$$

and

$$\mathfrak{C}^{\alpha_1\cdots\alpha_p\gamma}\mathfrak{L}_{\alpha_1\cdots\alpha_p;\,\lambda_1\cdots\lambda_{p-2}}=0)$$

and three quantities $\mathfrak{P}_{\mu\nu}{}^{\omega\beta\alpha_1\cdots\alpha_p}$, $\mathfrak{P}_{1\ \mu\beta\alpha_1\cdots\alpha_p}$, $\mathfrak{P}_{p-3}{}^{\beta\alpha_1\cdots\alpha_p\gamma_1\cdots\gamma_{p-3}}$ which are all symmetric in the indices $\beta\alpha_1\cdots\alpha_p$ and linear combinations of the quantities $\mathfrak{B}$ and $\mathfrak{C}$ (omitting indices), satisfying the relations

$$\mathfrak{P}_{\mu\nu}{}^{\omega\beta\alpha_1\cdots\alpha_p}a_{\lambda\alpha_1\cdots\alpha_p}=\mathfrak{A}\delta^{\beta}_{(\mu}\delta^{\alpha_1\cdots\alpha_p}_{\nu)\lambda}\delta^{\omega}_{\lambda},\tag{1}$$

$$\mathfrak{P}_{1\ \omega}{}^{\beta\alpha_1\cdots\alpha_p}a_{\lambda\alpha_1\cdots\alpha_p}=\mathfrak{A}\mathfrak{E}^{\beta\alpha_1}\mathfrak{e}_{\lambda\omega},\quad\text{etc.},$$

where we raise and lower indices by the unit skew-symmetric tensors $\mathfrak{E}^{\alpha\beta}$ and $\mathfrak{e}_{\alpha\beta}$ respectively. In §2 the object of affine connection in $F_2^{(p)}$ is defined completely by the condition $\mathfrak{P}_{\mu\nu}{}^{\omega\beta\alpha_1\cdots\alpha_p}\nabla_\beta a_{\alpha_1\cdots\alpha_p}=0$, that is,

$$\Gamma^{\omega}{}_{\mu\nu}=[1/p\mathfrak{A}]\mathfrak{P}_{\mu\nu}{}^{\omega\beta\alpha_1\cdots\alpha_p}\partial_\beta a_{\alpha_1\cdots\alpha_p}$$

making use of the relation (1). Then the covariant derivative of the fundamental tensor $\nabla_\beta a_{\alpha_1\cdots\alpha_p}$ can be expressed in a linear form in the three fundamental quantities S_μ, $\mathfrak{S}^{\lambda_1\cdots\lambda_{p-3}}$, $\mathfrak{T}^{\lambda_1\cdots\lambda_{p-3}}$ with coefficients composed of the fundamental tensor and the quantity $\mathfrak{L}$, where the quantities S, $\mathfrak{S}$ and $\mathfrak{T}$ are obtained by contracting

$$\frac{-1}{2p\mathfrak{A}}\mathfrak{P}_{1\ \mu}{}^{\beta\alpha_1\cdots\alpha_p},\qquad \frac{p-4}{(p-2)\mathfrak{A}}\mathfrak{B}^{\beta\alpha_1\cdots\alpha_p\lambda_1\cdots\lambda_{p-3}}$$

and

$$\frac{1}{\mathfrak{A}}\mathfrak{P}_{p-3}{}^{\beta\alpha_1\cdots\alpha_p\lambda_1\cdots\lambda_{p-3}}$$

with $\partial_\beta a_{\alpha_1\cdots\alpha_p}$ respectively. Therefore any differential comitant of order q of the fundamental tensor can be expressed by an algebraic comitant of the fundamental tensor, S, $\mathfrak{S}$, $\mathfrak{T}$ and their covariant derivatives of order $q-1$ with respect to the invariant connection. The relations $S=0$, $\mathfrak{S}=0$, $\mathfrak{T}=0$ are necessary and sufficient for $F_2^{(p)}$ to be flat, i.e., for there to exist a coordinate system in which $a_{\alpha_1\cdots\alpha_p}$ are all constants. Moreover, it is proved that $\mathfrak{S}=0$, $\mathfrak{T}=0$ are necessary and sufficient for existence of the connection $\bar{\Gamma}^{\alpha}{}_{\mu\nu}=\Gamma^{\alpha}{}_{\mu\nu}+2S_{[\mu}\delta_{\nu]}{}^{\alpha}$ for which $\bar{\nabla}_\beta a_{\alpha_1\cdots\alpha_p}=0$ and that $\mathfrak{S}=0$ is necessary and sufficient for existence of the connection

$$\bar{\Gamma}^{\alpha}{}_{\mu\nu}=\Gamma^{\alpha}{}_{\mu\nu}-\frac{2}{p}a^{\alpha}{}_{\mu\nu\lambda_1\cdots\lambda_{p-3}}\mathfrak{T}^{\lambda_1\cdots\lambda_{p-3}}$$

for which $\bar{\nabla}_{(\beta}a_{\alpha_1\cdots\alpha_p)}=0$.

The conformal transformation ${}^*a=\lambda a$ is considered in §3 and we have the conformal invariant quantities $\overset{(k)}{\mathfrak{B}}$, $\tilde{\mathfrak{S}}$, $\tilde{\mathfrak{T}}$ defined by $\mathfrak{E}^{\lambda\sigma}\nabla_\lambda S_\mu$, $|\mathfrak{A}|^{1/2(p-1)}\mathfrak{S}$, $|\mathfrak{A}|^{1/2(p-1)}\mathfrak{T}$ respectively. Concerning these fundamental quantities many theorems are proved. For example, the vanishing of these three quantities is necessary and sufficient for conformal flatness of $F_2^{(p)}$. A similar discussion is given in §4 in the space $\mathfrak{F}_2^{(p)}$ endowed with a field of pseudo-tensor $A_{\alpha_1\cdots\alpha_p}$ whose discriminant does not vanish. Since the space $\mathfrak{F}_2^{(p)}$ may be also considered the one endowed with a p-textile (p-Gewebe), some theorems concerning p-textiles are obtained. Finally, the general theory is applied to the special cases $p=3$, 4 in §5. Because $\mathfrak{S}=0$, $\mathfrak{T}=0$ for $p=3$ and $\mathfrak{S}=0$ for $p=4$, the theory becomes considerably simpler in these special cases.

A. Kawaguchi (Sapporo).

Moór, Arthur. Über die Dualität von Finslerschen und Cartanschen Räumen. Acta Math. **88**, 347–370 (1952).

It is the purpose of the present paper to find the relations between the fundamental quantities of a Finsler space and those of a Cartan space in the case that there exists the one-to-one correspondence:

$$\mu_i=\frac{1}{(g^*(x,\dot{x}))^{1/2}}g^*_{ik}(x,\dot{x})\dot{x}^k,\quad \dot{x}^i=(g(x,\mu))^{1/2}g^{ik}(x,\mu)\mu_k$$

between a line element $(x^i,\dot{x}^i)$ of the Finsler space and a hyperplane element (x^i,μ_i) of the Cartan space, where $g^*_{ik}(x,\dot{x})$ and $g^{ik}(x,\mu)$ are the metric tensors of the Finsler and Cartan spaces respectively (quantities with stars are of the Finsler space). g^* and g are the determinants of the metric tensors. First the following results are obtained: (1) $g^*_{ij}=g_{ij}$, $g^{*ij}=g^{ij}$, $g^*=g$; (2) the normal unit vector $l_i=(g/F)\mu_i$ coincides with the unit vector $l_i^*=\partial F^*/\partial\dot{x}^i$ and the torsion tensors $A^{*ijk}=A^{ijk}$; (3) $A^k=0$ and $A_i^*=0$. Then a Riemannian space osculating to the Cartan space along a hyperplane element family $(x^i(t),\mu_i(t))$ is defined analogously to that osculating to the Finsler space along a line element family $(x^i(t),\dot{x}^i(t))$ defined by O. Varga [Monatsh. Math. Phys. **50**, 165–175 (1941); these Rev. **5**, 218]. Under the assumption of the existence of such osculating Riemannian spaces, the author shows that the curvature tensor of the osculating Riemannian space is identical with the principal curvature tensor $T^*{}_{ijkl}$ of the Finsler space and with the curvature tensor $\bar{R}_{ijkl}$ of the Cartan space; hence $\bar{R}_{ijkl}=T^*{}_{ijkl}$ under the correspondence in consideration, and along the family of line or hyperplane elements the invariant differential of a vector is identical with the same in the osculating Riemannian space. Finally, it is shown that the relations $R^*{}_{ijkl}=0$ and $\bar{R}_{ijkl}=0$ are necessary for the existence of the osculating spaces along the families in the Finsler and Cartan spaces respectively.

A. Kawaguchi (Sapporo).

Craig, Homer V. On multiple parameter Jacobian extensors. Tensor (N.S.) **2**, 27–35 (1952).

In this paper, the author introduces his scheme for forming absolute matrix and Jacobian matrix extensors in the case of more than one independent variable. The basic ideas are contained in a new matrix superscript notation and in the definition of the transformation laws for matrix and Jacobian matrix extensors. After discussing the relation of his theory to the general theory of A. Kawaguchi [Monatsh. Math. Phys. **48**, 329–339 (1939); these Rev. **1**, 87] the author shows that: (1) the addition property of the order of differentiation is valid; (2) product differentiation follows

the generalized Leibnitz rule. In addition, the author shows how to reduce Jacobian matrix extensors to ordinary extensors and also how to construct Jacobian matrix extensors from weighed scalars and from absolute matrix extensors. Finally, certain basic matrix extensors are related to contracted Christoffel symbols and intrinsic derivatives.

N. Coburn (Ann Arbor, Mich.).

Kawaguchi, Michiaki, Jr. A generalization of the extensor. Tensor (N.S.) **2**, 59–66 (1952).

In the present article, the theory of extensors which are functions of a single parameter in m-dimensional space is considered. To obtain generalized extensors (g-extensors), the author defines a generalized extended point transformation by considering the totality of the transformation laws of the various partial derivatives of a scalar expoint function. The inverse transformation is shown to exist. Further, it is shown that g-extensors consist of classes of reduced extensors. An example of a g-contravariant exvector is given and the relation of the theory to that of Finsler space is indicated. *N. Coburn* (Ann Arbor, Mich.).

Kawaguchi, Michiaki, Jr. On a generalization of the multiple parameter extensor. Tensor (N.S.) **2**, 99–101 (1952).

A generalization analogous to that of the paper reviewed above is carried out for the case of the multiple parameter theory of H. V. Craig [see the paper reviewed second above]. It is indicated that the new g-extensors consist of classes of reduced extensors. *N. Coburn* (Ann Arbor, Mich.).

Sasayama, Hiroyoshi. On the exterior differential forms on higher order manifolds. Tensor (N.S.) **2**, 36–46 (1952).

The author studies p-fold multiple integrals of Mth order. Conditions such that a multiple integral remain invariant under parameter and coordinate transformations are obtained. A generalized Stokes theorem for these integrals is established. The integrand which appears in these integrals suggests the definition of exterior differential forms of higher order and rank. The addition, exterior multiplication, and differentiation of these forms are defined and additional results on their algebra and calculus derived.

A. Fialkow (Brooklyn, N. Y.).

Hawley, N. S. Constant holomorphic curvature. Canadian J. Math. **5**, 53–56 (1953).

This paper gives a number of properties (in the large) of a complex manifold with Kähler metric and constant holomorphic curvature, i.e., such that the curvature tensor is of the form $R_{a\bar{a}c\bar{b}} = -\frac{1}{2}b(g_{a\bar{b}}g_{c\bar{a}} \pm g_{a\bar{a}}g_{c\bar{b}})$ where b is a constant. A proof is given of the main theorem (due to Bochner) which says that if the space is complete, its universal covering space is analytically isometrically equivalent to a euclidean cell if $b=-2$ and to a complex projective space if $b>0$. *A. G. Walker* (Liverpool).

Blanchard, André. Recherche de structures analytiques complexes sur certaines variétés. C. R. Acad. Sci. Paris **236**, 657–659 (1953).

A variety V_{2n} is almost complex if there exists on it a tensor field J_i^j satisfying the equation $J_i^j J_j^k = -S_i^k$. The structure defined by J_i^j is complex if J_i^j (a) satisfies the Eckmann-Frölicher condition [same C. R. **232**, 2284–2286 (1951); these Rev. **13**, 75], (b) is real-analytic. It is shown that if V_{2n} is a domain of R_{2n}, and J_i^j is compatible with the Euclidean metric of R_{2n}, condition (a) is sufficient. It is further shown that if V_{2n} is a conformally flat Riemannian space, the fibre bundle B on V_{2n} whose fibre is $SO(2n)/U(n)$ and whose associated principal bundle is the bundle of $2n$-forms of V_{2n} has a complex structure naturally related to the metric of V_{2n}, and that the complex structures of V_{2n} compatible with its metric correspond to the cross-sections of B which are complex submanifolds of B. As a corollary, it is shown that a $2n$-space has no complex structures compatible with its natural metric if $n>1$.

W. V. D. Hodge (Cambridge, England).

NUMERICAL AND GRAPHICAL METHODS

***Hartree, D. R. Numerical analysis.** Oxford, at the Clarendon Press, 1952. xiv+287 pp. $6.00.

The author draws on his extensive experience to write a treatise on "practical methods for obtaining numerical results". His emphasis on the practical aspects of the subject has three important implications concerning the general character of the book. (1) In general, for each type of problem a single method is presented in detail, with examples, as the one which the author considers usually most convenient, and alternative procedures are discussed more briefly, with reasons for their being less useful. Obviously such a selection is to some extent subjective and may depend on the nature of the problems with which the author has had most experience. Nevertheless, such an approach is to be commended, because one of the difficulties with which the inexperienced user of numerical procedures is confronted is the multiplicity of methods available. (2) The choice of the most practical method, or, indeed, of what is practical and what is not, will depend very strongly on the computing equipment available. The present book is written for computation by a desk calculator. The choice of methods would have been different in many cases if the work were to be done by an automatic high speed machine. The author does, however, conclude his book with a brief but illuminating chapter on the theory of programming and coding problems for the latter type machines, especially the British EDSAC. This chapter covers most of the main devices useful in programming, including the use of subroutines, but omits the difficult problem of scaling. (3) Theoretical considerations are largely omitted or subordinated to practical convenience. An exception is a presentation of W. E. Milne's general method for the estimation of truncation errors. No attempt has been made to show that certain formulas may be theoretically optimum. In a few places the essentially formal nature of the treatment has led to a lack of precision of statement. For instance, from the statements on page 50 one might infer that a function which possesses derivatives of all orders must have a Taylor's series representation. Again, on page 35 it is not correct that an odd function with period 2π whose nth derivative has a mean square value which is bounded for all n must be a constant multiple of $\sin x$. The author's argument breaks down if some derivative fails to exist at a single point, but such a failure would not affect the mean square value. It is not clear to the reviewer that such possibilities have been excluded, a saw-tooth function for example.

A noteworthy feature of the book is the systematic use of the operator notation for differences, following the ideas suggested by J. G. L. Michel and W. G. Bickley. The use of formal expansions in these operators is a convenient unifying device. In choosing methods, the author generally prefers formulas in terms of differences rather than in terms of function values. In particular, he objects to the use of the Lagrange formula for inverse interpolation and of Simpson's rule for quadrature.

After a general introduction and description of desk calculating machines, there are chapters on differences, interpolation, integration and differentiation, and integration of ordinary differential equations with either one or two point boundary conditions, followed by a treatment of systems of linear equations, matrix inversion, and characteristic value determination, then chapters on non-linear algebraic equations including simultaneous systems of such, finite differences in two or more variables with the solution of boundary value problems in partial differential equations by relaxation and other methods, and finally a chapter on miscellaneous processes such as summation of series, harmonic analysis, and smoothing. The book is described by the author as an introduction to the subject, but it should be invaluable as a practical guide to the computer.

P. W. Ketchum (Urbana, Ill.).

Blanch, Gertrude, and Yowell, Everett C. Addendum to a guide to tables on punched cards. Math. Tables and Other Aids to Computation **7**, 1–6 (1953).
See same journal **5**, 185–212 (1951); these Rev. **13**, 495.

Domb, C. Tables of functions occurring in the diffraction of electromagnetic waves by the Earth. Advances in Physics **2**, 96–102 (1953).

This paper gives tables of the functions $f_n(y)$ which are needed in the expansion given by Pryce [Advances in Physics **2**, 67–95 (1953); these Rev. **14**, 703]. The functions $f_n(y)$ are essentially Airy functions of complex argument. They are given by the equation

$$f_n(y)=A_i(-a_n+\exp(\pi i/3)y)/\exp(\pi i/3)A_i'(-a_n)$$

where the numbers $-a_n$ are the zeros of the Airy integral.

B. Friedman (New York, N. Y.).

Goodell, John D. The foundations of computing machinery. I. J. Computing Systems **1**, 1–13 (1952).
Goodell, John D. The foundations of computing machinery. II. J. Computing Systems **1**, 86–110 (1953).

In these two papers the author is interested in establishing rapport between the mathematical logician and the computing machine engineer. To this end he introduces a number of logical notations which might be useful for machine design. These notations are pictorial in character and represent such operations as "and", the conjunctive and disjunctive "or", "not", etc. Using these symbols, one can design circuits in a logical sense. This approach is similar to von Neumann's use of the Pitts-McCulloch "neural" notation. The author considers some of these circuits in detail to illustrate the method. *H. H. Goldstine.*

Lehmer, D. H. The sieve problem for all-purpose computers. Math. Tables and Other Aids to Computation **7**, 6–14 (1953).

Wilkes, M. V., and Stringer, J. B. Micro-programming and the design of the control circuits in an electronic digital computer. Proc. Cambridge Philos. Soc. **49**, 230–238 (1953).

Blanc, Charles. Etude stochastique de l'erreur dans un calcul numérique approché. Comment. Math. Helv. **26**, 225–241 (1952).

Following ideas announced earlier [C. R. Acad. Sci. Paris **233**, 683–684, 726–727 (1951); these Rev. **13**, 368], the author develops his stochastic approach to the accuracy with which one linear functional $L_1(f)$ (e.g., $\int_a^b f(t)dt$) is approximated by a second linear functional (e.g., the trapezoidal formula $\frac{1}{2}[f(a)+f(b)]$). Rather than bound $|L_1(f)-L_2(f)|$ (e.g., in terms of $\max_{a\le t\le b}|f''(t)|$), the author measures the variance of the random variable $L_1(\xi)-L_2(\xi)$, where $\xi(t)$ is a random function assumed to possess a covariance $r(t,t')=E[\xi(t)\cdot\xi(t')]$. It is assumed that $L_1(\xi)-L_2(\xi)$ has zero mean.

Formal expressions for $E[L_1(\xi)-L_2(\xi)]^2$ in terms of $r(t,t')$ are given for selected formulas of interpolation, finite-differencing, numerical quadrature, and linear representation of a function in an incomplete basis. The author also treats the truncation error resulting from difference approximations to differential equations, particularly the Dirichlet problem in two dimensions. There are two numerical examples based on $r(t,t')$ of type $A\sin a(t-t')/a(t-t')$, the covariance of a stationary random function with a uniformly distributed spectrum in the finite band $(-a,a)$. The methods apply also in n dimensions. *G. E. Forsythe.*

Ginzburg, B. L. Generalization of various interpolation formulas to the case of unequal intervals. Akad. Nauk SSSR. Inženernyĭ Sbornik **12**, 201–220 (1952). (Russian)

This paper gives formulas for interpolation, numerical differentiation, and numerical integration expressed by means of divided differences. Using a notation for divided differences somewhat like Sheppard's notation for central differences in the case of equally spaced abscissas the author obtains for unequally spaced abscissas the generalizations of Newton's two interpolation formulas. By changes of notation (based essentially on Sheppard's rules) he goes on to get the generalizations of Gauss's, Stirling's, Bessel's, and Everett's formulas. From these in turn he gets formulas for numerical differentiation and integration in terms of divided differences. *W. E. Milne* (Corvallis, Ore.).

Obmoršev, A. N. Graphical solution of a characteristic equation with complex coefficients. Akad. Nauk SSSR. Inženernyĭ Sbornik **13**, 190–192 (1952). (Russian)

The general polynomial equation is written in the form $F(z)+az+b=0$, where $z=x+iy$. Letting $F(z)=U+iV$ and $-az-b=u+iv$, the roots of the equation are found from the intersections of the surface $U(x,y)$ with the plane $u(x,y)$ and the surface $V(x,y)$ with the plane $v(x,y)$. For graphical construction, the surfaces and planes are reduced to families of curves and straight lines by taking a series of fixed values of y. The locus of the intersections of corresponding curves and lines is the desired intersection of the surface and the plane. The intersections of the two loci give the desired roots of the given polynomial equation. The method is illustrated for the case of a quartic equation for which the graphs and the numerical results are shown.

M. Goldberg (Washington, D. C.).

van der Heiden, J. A. Method of least squares applied to a set of simultaneous equations which have one or more constants in common. Statistica, Rijswijk **6**, 107–112 (1952). (Dutch. English summary)

Unger, H. Orthogonalisierung (Unitarisierung) von Matrizen nach E. Schmidt und ihre praktische Durchführung. Z. Angew. Math. Mech. **31**, 53–54 (1951).

Let the square matrix $\mathfrak{A}$ consist of rows $\mathfrak{a}_1, \cdots, \mathfrak{a}_n$ of complex elements. In the Schmidt unitarization process one successively determines orthogonal rows $\mathfrak{r}_1, \cdots, \mathfrak{r}_n$ by the formulas

$$\mathfrak{r}_i = -\sum_{p=1}^{i-1} l_{ip}\mathfrak{r}_p + \mathfrak{a}_i \quad (i=1, \cdots, n), \tag{1}$$

where $l_{ip}=\mathfrak{a}_i\mathfrak{r}_p^*/\mathfrak{r}_p\mathfrak{r}_p^*$ and the star denotes conjugate transpose. A number of variations of the process are discussed, together with its application to the solution of a system of linear equations $\mathfrak{A}\mathfrak{x}=\mathfrak{y}$.

The principal observation is that (1) involves a great many operations with complex numbers, whereas the following method can yield the $\mathfrak{r}_i$ in fewer operations, many of which are with real numbers. Let $\mathfrak{L}$ be the matrix of l's in (1), with $l_{ii}=1$ and $l_{ij}=0$ $(j>i)$. Let $\mathfrak{R}$ be the matrix of rows $\mathfrak{r}_1, \cdots, \mathfrak{r}_n$. Then $\mathfrak{L}\mathfrak{R}=\mathfrak{A}$, whence $\mathfrak{R}=\mathfrak{L}^{-1}\mathfrak{A}$. The author's proposal is first to compute the hermitian matrix $\mathfrak{A}\mathfrak{A}^*$, and then to carry out the Cholesky triangularization scheme [see R. Zurmühl, Matrizen, Springer, Berlin-Göttingen-Heidelberg, 1950, pp. 248–262; these Rev. **12**, 73] on the system $(\mathfrak{A}\mathfrak{A}^*)\mathfrak{X}_1=\mathfrak{A}$. Since $\mathfrak{R}\mathfrak{R}^*=\mathfrak{L}^{-1}\mathfrak{A}\mathfrak{A}^*(\mathfrak{L}^*)^{-1}=\mathfrak{D}^2$, a real diagonal matrix, the triangular decomposition of $\mathfrak{A}\mathfrak{A}^*$ is $\mathfrak{L}\mathfrak{D}^2\mathfrak{L}^*$. Hence the Cholesky process yields

$$\mathfrak{L}^{-1}(\mathfrak{A}\mathfrak{A}^*)\mathfrak{X}_1=\mathfrak{L}^{-1}\mathfrak{A}=\mathfrak{R},$$

as asserted. *G. E. Forsythe* (Los Angeles, Calif.).

Filipowsky, Richard. Numerical calculations in electrical engineering and electronics. I. Calculation of determinants of higher order and the solution of simultaneous algebraic equations. J. Madras Inst. Tech. **1**, 64–88 (2 plates) (1952).

The author urges electrical engineers to learn a universal elimination algorithm for solving the problems of the title. He describes the unabridged Gauss process, and give a computation form suitable for up to $n=6$ unknowns. He advises carrying n guard digits unless the answer will be improved from the residual equations. Rather interesting is the advocacy of rough elimination with a slide-rule, with successive approximations to the answer being obtained by solving the residual equations on the basis of the rough elimination. There are numerical examples.

One misses the abridged elimination methods, which are much faster when desk calculators are available. The reviewer doubts the validity of the algorithm (p. 88) proposed to accelerate the convergence of the successive approximations. *G. E. Forsythe* (Los Angeles, Calif.).

Sassenfeld, H. Ein hinreichendes Konvergenzkriterium und eine Fehlerabschätzung für die Iteration in Einzelschritten bei linearen Gleichungen. Z. Angew. Math. Mech. **31**, 92–94 (1951).

The author gives two improvements of Collatz's [Z. Angew. Math. Mech. **22**, 357–361 (1942); these Rev. **5**, 50] sufficient condition, $\sum_{k=1}^{n}|a_{ik}|<2|a_{ii}|$ (all i), for the convergence of the single-step iteration for solving a system of linear equations $\sum_{k=1}^{n}a_{ik}x_k+a_i=0$. One of the new conditions is that $\max_i \alpha_i<1$, where the α_i are recursively defined by $|a_{ii}|\alpha_i=\sum_{k=1}^{i-1}|a_{ik}|\alpha_k+\sum_{k=i+1}^{n}|a_{ik}|$. Each condition leads also to a bound on the speed of convergence. There is a numerical example. *G. E. Forsythe.*

Krasnosel'skiĭ, M. A., and Kreĭn, S. G. An iteration process with minimal residuals. Mat. Sbornik N.S. **31(73)**, 315–334 (1952). (Russian)

Let B be a real positive definite matrix. The authors introduce the "α-processes," a family of gradient methods for solving a system of linear equations $Bx=b$, depending on a real parameter α. Let x_0^α be arbitrary. For each $k=0, 1, \cdots$, a sequence $\{x_k^\alpha\}$ converging to $x^*=B^{-1}b$ is defined by letting $x_{k+1}^\alpha=x_k^\alpha-c_\alpha\Delta_k^\alpha$, where $\Delta_k^\alpha=Bx_k^\alpha-b$, and where $c_\alpha=(B^\alpha\Delta_n^\alpha, \Delta_n^\alpha)/(B^{\alpha+1}\Delta_n^\alpha, \Delta_n^\alpha)$. (In x_k^α and Δ_k^α, α is a superscript; but B^α is the αth power of B.)

For real γ, let $\|z\|_\gamma$, the "γ-length" of z, be $(z, B^\gamma z)^{\frac{1}{2}}$. Then the α-process selects x_{k+1}^α among all vectors of the form $x_k^\alpha-\gamma\Delta_k^\alpha$ $(-\infty<\gamma<\infty)$ so as to minimize $\|x_{k+1}^\alpha-x^*\|_{\alpha+1}=\|\Delta_{k+1}^\alpha\|_{\alpha-1}$. By a very simple argument it is shown that all α-processes have the same norm, i.e., one always has

$$\|\Delta_{k+1}^\alpha\|_{\alpha-1}\leqq\left(\frac{M-m}{M+m}\right)\|\Delta_k^\alpha\|_{\alpha-1},$$

where m, M are the least, greatest eigenvalues of B, and equality is attained for some x_k^α. (Details are missing on the matter of equality.) In a preliminary theorem it is shown that $\log(B^\alpha x, x)$ is a convex function of α.

The 0-process is the method of steepest descent of Kantorovich and others, while the 1-process is proposed for practical use and given the name of the title. The choice between the two depends on what measure of the error one is striving to reduce. In addition to the minimizing nature of the 0- and 1-processes inherent in the above, it is shown that the 0-process is the best of the "practically realizable" α-processes (i.e., $\alpha=0, 1, 2, \cdots$) in the one-step reduction of $\|x_k-x^*\|_0$. The non-linear transformation L: $x_k^1\to x_{k+1}^1$ is studied in some detail; it is shown that L is commonly $n-1$ to 1. *G. E. Forsythe* (Los Angeles, Calif.).

Brodskiĭ, M. L. Probability estimates of errors in the determination of the characteristic values and characteristic vectors of a varied matrix. Uspehi Matem. Nauk (N.S.) **7**, no. 5(51), 205–214 (1952). (Russian)

Let the matrix A of order n have a simple characteristic value λ with corresponding characteristic vector u and adjoint characteristic vector u^*; thus $Au=\lambda u$, $A^*u^*=\bar{\lambda}u^*$ [where $A^*=\bar{A}'$]. If A is changed to $A+\delta A$, it is desired to bound the new values $\lambda+\delta\lambda$ and $u+\delta u$. By differentiation the author finds $\delta\lambda=(\delta A\, u, u^*)/(u, u^*)$, and gets the absolute bound

$$|\delta\lambda|\leqq\epsilon\sum_{i,k=1}^{n}|u_i||u_k^*|/|(u, u^*)|=\epsilon O(n),$$

as $n\to\infty$. If the δa_{ik} are independent random variables with dispersions $D(\delta a_{ik})\leqq\sigma^2$ (all i, k), then

$$D(\delta\lambda)\leqq\sigma^2\|u\|^2\|u^*\|^2|(u, u^*)|^{-2}.$$

In case $u=u^*$ (as for hermitian A), $D(\delta\lambda)\leqq\max D(\delta a_{ik})$, and, with high probability, $|\delta\lambda|\leqq\max_{i,k}|\delta a_{ik}|=\epsilon$. Similar absolute and probabilistic bounds are worked out for the components of u.

Absolute and probabilistic bounds for $\delta\lambda$ and δy_i are computed in great detail for one case of dependent a_{ij}, a simple finite-difference approximation to a general one-dimensional Sturm-Liouville problem $(d/dx)(Rdy/dx)-Py=\lambda y, R(x)>0$, with $y(0)=y(1)=0$. The difference equation $h^{-2}\Delta^2y_i=\lambda y_i$ is treated numerically. *G. E. Forsythe.*

Synge, J. L. A simple bounding formula for integrals. Canadian J. Math. **5**, 46–52 (1953).

The formula is $\int_D F \cdot dD = (\Delta/(n+1))\sum m_i F_i + E_1 + E_2$, where D is a domain of Euclidean n-space and F a real function in D, continuous and with continuous derivatives up to second order. A subset $T \subset D$ is properly divided into congruent n-simplices (cells with $n+1$ vertices) with n-volume Δ. Here F_i is the value of F at vertex i, and m_i is the number of cells meeting there. E_1 is an error term, bounded by $|E_1| \leq (D-T)\bar{F}$ with $\bar{F}$ as upper bound for $|F|$ in D, and E_2 is another error term, bounded by $|E_2| \leq kMT$. Here M is an upper bound for $|d^2F/ds^2|$ for all points of D and all directions s; $k = \frac{1}{2}(r_0^2 - P/\Delta)$ is a cell-constant, referring to the radius r_0 of the circumscribed sphere of a cell and to the integral $P = \int_\Delta r^2 dD$ with r as distance from the circumcentre. Proofs are of remarkable simplicity. A comparison with another formula given by R. v. Mises concludes the paper.

H. Bückner (Minden).

Löwdin, Per-Olov. Approximate formulas for many-center integrals in the theory of molecules and crystals. J. Chem. Phys. **21**, 374–375 (1953).

Fox, L. The use of large intervals in finite-difference equations. Math. Tables and Other Aids to Computation **7**, 14–18 (1953).

Cahen, Gilbert. Étude topologique de certaines équations différentielles non linéaires. C. R. Acad. Sci. Paris **235**, 1003–1005 (1952).

A graphical construction for the trajectories of

$$\ddot{x} + g(x)\dot{x} + r(x) = 0$$

is described.

G. E. H. Reuter (Manchester).

Kulikov, N. K. Approximate determination of the period of free oscillations of a nonlinear system with one degree of freedom. Akad. Nauk SSSR. Inženernyĭ Sbornik **13**, 187–189 (1952). (Russian)

Consider the oscillations described by the equation $\ddot{x} + F(x) = 0$, where $F(x)$ is continuous and monotonely increasing, $F(0) = 0$. By using an arithmetic average the author derives the approximation $\pi((a/F(a))^{1/2} + (b/F(b))^{1/2})$ for the period of the oscillations between the extreme positions $x = a > 0$ and $x = b < 0$. A comparison with some accurately known results shows good agreement.

M. Golomb.

Myklestad, N. O. Numerical analysis of forced vibrations of beams. J. Appl. Mech. **20**, 53–56 (1953).

Iino, Riichi. A method for solving non-linear partial differential equation. Mem. Fac. Sci. Eng. Waseda Univ. **16**, 107–108 (1952).

The author extends to the case of partial differential equations a method of Pipes [J. Appl. Phys. **23**, 202–207 (1952); these Rev. **13**, 783] for the solution of a special class of non-linear differential equations. The method is illustrated by a simple example.

W. E. Milne.

Fehlberg, Erwin. Bemerkungen zur numerischen Behandlung des Dirichletschen Problems für spezielle Ränder. Acta Math. **87**, 361–382 (1952).

The author discusses the problem of solving numerically the equation $u_{xx} + Au_{yy} = e(x, y)$, given the boundary values of u on the square $|x| = 1$, $|y| = 1$ or on the circle $x^2 + y^2 = 1$. The function $u(x, y)$, $e(x, y)$ and the boundary values are expanded in a double series of products of Legendre polynomials $P_m(x)P_n(y)$ and then by comparing coefficients a set of linear equations is obtained. The author gives explicit formulas for the coefficients of u in case the expansions end with $P_3(x)P_3(y)$.

B. Friedman (New York, N. Y.).

Motzkin, T. S., and Wasow, W. On the approximation of linear elliptic differential equations by difference equations with positive coefficients. J. Math. Physics **31**, 253–259 (1953).

The problem considered is that of approximating a given linear elliptic second-order partial differential equation $L[u] = 0$ in n-space by a linear difference equation $L_h[u] = 0$ (referring to a square mesh of width h) in such a way that (i) the operator $L_h[u]$ should be a formal analogue of $L[u]$, (ii) the equation $L_h[u] = 0$ should express the value of u at a lattice point P as a weighted average, with positive weights, of the values of u at certain lattice points $Q_1, \cdots, Q_s$. If the system of neighboring points $S = \{Q_1, \cdots, Q_s\}$ is given and contains at least all lattice points whose distance from P does not exceed $(nh)^{1/2}$, condition (i) can always be satisfied. The authors prove that for no S can condition (ii) be satisfied for all elliptic equations, but that for any given equation condition (ii) can be satisfied by choosing S appropriately. Condition (ii) is of importance for proving the solvability of Dirichlet's problem for the difference equation and the convergence of the solution of this problem to the appropriate solution of the differential equation, and also in view of the connection between linear elliptic equations and random walks. Condition (ii) also implies a maximum principle for solution of $L_h[u] = 0$, which the authors establish in the last section of this paper.

L. Bers (New York, N. Y.).

van den Dungen, F. H. Formules pour l'intégration numérique de l'équation des ondes. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 669–684 (1952).

The author considers further the problem of integrating the wave equation $\Delta u = u_{tt}$ given $u = f(x, y)$ and $u_t = g(x, y)$ at $t = 0$. In a previous paper [same Bull. (5) **38**, 39–49 (1952); these Rev. **14**, 93] he has given a method which is essentially that of expanding u in a Taylor's series in t. In this paper he points out that the finite-difference approximations to the derivatives which he used previously causes the x- and y-axis to have a privileged role which is not warranted by the problem. He, therefore, suggests approximating the derivatives by the use of circular means. For example, if f has sufficiently many continuous derivatives and if m is the mean value of f on a circle center the origin and radius kh, then

$$m = f_0 + \frac{h^2k^2}{4}\Delta f_0 + \frac{3}{2}\frac{h^4k^4}{4!}\Delta^2 f_0 + \cdots.$$

B. Friedman (New York, N. Y.).

Woods, L. C. The numerical solution of two-dimensional fluid motion in the neighbourhood of stagnation points and sharp corners. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2726** (12,887), 12 pp. (1952).

Nets of streamlines and potentials are calculated by an adaptation of the relaxation method. The main ideas are: (i) algebraic expansion of the logarithmic singularity near a sharp corner, and (ii) an iterative treatment, following Thom [same Rep. and Memoranda no. **1604** (1934)], of the boundary conditions in the (ϕ, ψ)-plane.

G. Birkhoff.

Horton, G. K., and Sharp, R. T. Approximate wave functions for unbound relativistic particles in a Coulomb field. Physical Rev. (2) **89**, 885–886 (1953).

Mikeladze, Š. E. On approximate solution of integral equations of Volterra type. Akad. Nauk Gruzin. SSR. Trudy Mat. Inst. Razmadze **18**, 315–325 (1951). (Russian. Georgian summary)

After a short section listing the numerical integration formulas to be used, there is a section wherein Volterra's linear integral equation

$$\phi(x)=f(x)+\lambda\int_a^x K(x,s)\phi(s)ds$$

is replaced by a system of linear algebraic equations with unknowns $\phi(a+nh)$. More interesting is the third section, in which the nonlinear equation is solved by a method resembling Milne's method for solution of an ordinary linear differential equation. The values $\phi(a+h)$, $\phi(a+2h)$, $\cdots$, $\phi(a+nh)$ having been determined, $\phi[a+(n+1)h]$ is estimated with a Steffensen open-type formula and this estimate is improved with a closed-type formula. A shorter discussion of a similar numerical procedure for finding characteristic values and functions follows. This article is entirely expository, reference being made to other works of the author for proofs and derivations. *R. E. Gaskell* (Seattle, Wash.).

Biermann, L., und Billing, H. Die Göttinger elektronischen Rechenmaschinen. Z. Angew. Math. Mech. **33**, 48–60 (1953). (English, French and Russian summaries)

Brooker, R. A., and Wheeler, D. J. Floating operations on the EDSAC. Math. Tables and Other Aids to Computation **7**, 37–47 (1953).

Erismann, Th. Anciens et nouveaux appareils mécaniques d'intégration. Bull. Tech. Suisse Romande **79**, 45–48, 61–64 (1953).

Many, Abraham, Oppenheim, Uri, and Amitsur, Shimshon. An electrical computer for the solution of linear simultaneous equations. Rev. Sci. Instruments **24**, 112–116 (1953).

Scherberg, Max G., and Riordan, John F. Analogue calculation of polynomial and trigonometric expansions. Math. Tables and Other Aids to Computation **7**, 61–65 (1953).

Volkov, E. A. A mechanical apparatus for the solution of Poisson's equation and certain other equations of elliptic type. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk **1950**, no. 10, 3–17 (1950). (Russian)

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Iglisch, Rudolf, und Tietz, Horst. Die Kinematik des starren Körpers. Math.-Phys. Semesterber. **3**, 87–89 (1953).

Euler's equation in kinematics is derived by means of vector algebra.

Litvin, F. L. Determination of tooth thickness of worms and helical gears by means of rollers and balls. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **10**, no. 39, 22–55 (1951). (Russian)

The necessary accuracy of tolerance control for spatial gears makes direct measurements of tooth thickness w_∞ impracticable. A method using cylindrical rollers or balls inserted in between the teeth is proposed. A (parametric) relation between the position parameter r of the ball or roller and w_∞ is developed but the equation is too involved for practical use. Therefore, approximate formulas for the relation between dr and dw_∞ are derived, which make it possible to estimate the relation of the error to the tolerance. Numerical examples are given for the three cases of evolute, convolute, and Archimedean tooth surfaces.

A. W. Wundheiler (Chicago, Ill.).

Budyka, Yu. N. Meshing theory and the relative wear resistance of plane engagements of general type. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **10**, no. 39, 56–74 (1951). (Russian)

The problem is to design open (nonlubricated) plane gearings with less than the conventional wear of cycloidal teeth. The author first determines the normal reaction between any two conjugate profiles (it is proportional to $(\sin 2\alpha_0)^{-1}$ where α_0 is the meshing angle. Since $\alpha_0 \neq 45°$ (for reason of undercutting), the author proposes to minimize wear by minimizing the specific work ξ of friction, and finds that ξ is not a minimum for cycloidal gearing. Some design aspects are discussed, and a continuation of the paper announced. *A. W. Wundheiler* (Chicago, Ill.).

Gallissot, François. Sur la discussion des éventualités dans la dynamique des solides en contact ponctuel avec frottement dans le cas où certaines vitesses de glissement sont nulles à l'instant initial. C. R. Acad. Sci. Paris **235**, 1471–1472 (1952).

Utilisant la méthode d'une note précédente [mêmes C. R. **235**, 937–939 (1952); ces Rev. **14**, 507] l'auteur discute les éventualités possibles à l'instant t_0 pour un ensemble de solides ayant p contacts entre eux. *J. Haantjes.*

Aržanyh, I. S. Conditions of applicability of a potential method of integration of the equations of motion of nonholonomic conservative systems. Doklady Akad. Nauk SSSR (N.S.) **87**, 15–18 (1952). (Russian)

Consider a holonomic conservative system, and take its equations of motion in the canonical form. Substitute $p_i=p_i(t, q_1, \cdots, q_n)$, $i=1, 2, \cdots, n$, in the canonical potential $H(t; q_1, \cdots, q_n; p_1, \cdots, p_n)$ to obtain $\bar{H}(t; q_1, \cdots, q_n)$. Then the equations of motion are satisfied identically if one puts $p_i=\partial W/\partial q_i$, $\partial W/\partial t+\bar{H}=0$. From this property of the Hamilton-Jacobi method, namely, of possessing a potential W for the field of impulses $p_i=p_i(t; q_1, \cdots, q_n)$, the author derives the name of potential method.

Given a nonholonomic conservative system, assume for simplicity that the equations of constraints as well as the kinetic potential L do not depend explicitly on time t. The author shows that the necessary and sufficient condition for applicability of the potential method is that a certain system of partial differential equations for W be consistent [see also Chaplygin, Collected works, vol. I, Gostehizdat, Moscow-Leningrad, 1948, pp. 15–25; and Aržanyh, Doklady Akad. Nauk SSSR (N.S.) **65**, 809–811 (1949); these

Rev. 10, 749]. Some possible ways of discussing the consistency are indicated. *E. Leimanis* (Vancouver, B. C.).

Dedecker, Paul. Sur le théorème de la circulation de V. Bjerknes. Mém. Inst. Roy. Météorol. Belgique **48**, 4 pp. (1951).

By applying results of D. C. Lewis and A. Lichnerowicz on invariant integrals of dynamical systems, the author obtains a generalization of the Bjerknes circulation theorem, in which the curve need not be material, and the time plays no privileged role. The "circulation" is now the line integral of $u_i dx_i - \frac{1}{2}u^2 dt$ around a closed circuit. The difference between the circulations about two circuits upon a single tube of trajectories in space-time is expressed in terms of an integral over a three-dimensional region. No hydrodynamnical interpretation for the result is given. *C. Truesdell.*

Urbański, W. S. Une méthode pour calculer les trajectoires des projectiles. Arch. Méc. Appl., Gdańsk **3**, 217–224 (1951). (Polish. French summary)

Consider the projectile as a material point moving in still air, and assume that the only forces acting are the gravity $m\mathbf{g}$ and the air resistance $m\mathbf{R}$, the latter acting in the direction of the tangent to the trajectory and opposite to the velocity $\mathbf{v}$ of the projectile. If R depends on the velocity v only and is proportional to the nth power, then $\mathbf{R}=-cF(v)\mathbf{v}/v$, where $cF(v)=bv^n$.

Instead of the variables v, x, y, t the author introduces a reduced velocity $p=v(b/g)^{1/n}$, reduced coordinates $X=(b^2g^{n-2})^{1/n}x$, $Y=(b^2g^{n-2})^{1/n}y$ and a reduced time $\theta=(bg^{n-1})^{1/n}t$, and gives the solution of the problem for the reduced variables in terms of τ and a parameter q, where τ is the angle of inclination to the horizontal of the tangent to the trajectory, and $q=u(b/g)^{1/n}$, u being the velocity of the particle at a summit of the trajectory. Since the trajectories corresponding to one and the same value of the parameter q are similar, it is sufficient for the solution of the particle problem to calculate a family of solutions for a certain range of values for q (separately for $n=2$, 3, 5). In the case where the density of the air varies with the altitude, or the exponent n varies with the velocity v, the ballistic coefficient b becomes a variable quantity and the method of successive arcs should be applied. In any case, the author claims that the procedure involved in his method is simpler than that of the G. H. M. (Garnier-Haag-Marcus) method. In the case of aviation bombs it is sufficient to restrict n to the case $n=2$. *E. Leimanis* (Vancouver, B. C.).

Leimanis, E. Über die Bewegung eines Massenpunktes in einem widerstehenden Mittel veränderlicher Dichte. Acta Physica Austriaca **6**, 319–321 (1953).

The author remarks that Th. Sexl [same Acta **5**, 148–151 (1951); these Rev. **14**, 101] considered the motion under constant gravity of a particle falling with air-density varying exponentially with altitude, and with retardation otherwise proportional to the nth power of the velocity. In particular, Sexl obtained closed formulas for the distance fallen, in the special cases $n=1$ and $n=2$ (important as ballistic approximations). This author considers the same problem altered by introducing gravity as an arbitrary known (integrable) function of the altitude. He obtains ingeniously a closed formula in the case $n=-2$. The reviewer questions the physical interest in motion with this choice of exponent.

A. A. Bennett (Providence, R. I.).

Hydrodynamics, Aerodynamics, Acoustics

***Troskolański, Adam Tadeusz. Hydromechanika techniczna. Tom I. Hydromechanika racjonalna. [Technical hydromechanics. Vol. I. Rational hydromechanics.]** Państwowe Wydawnictwa Techniczne, Warsaw, 1951. xx+352 pp. Zł. 40.

A textbook of hydrodynamics for engineering students. Table of contents: Introduction; Hydrostatics; Dynamics of ideal fluids; Dynamics of real fluids. A useful appendix provides the Russian, English, French, and German equivalents of Polish technical terms.

Huron, R. Contribution à l'étude de l'unicité des solutions du problème de représentation conforme de Helmholz. Ann. Fac. Sci. Univ. Toulouse (4) **15**, 5–78 (1951).

In this long paper on the classical free boundary problem, the author proves the uniqueness of an infinite cavity (he still refers to the cavity as "dead" water) developed behind an obstacle immersed in a uniform stream in a channel with parallel straight walls. His result improves that already given by Kravtchenko [J. Math. Pures Appl. (9) **20**, 235–303 (1941); these Rev. **4**, 58] only in allowing corners on the obstacle. The proof is the standard one evolved by Friedrichs, Leray, and Weinstein. In a final section, it is proved that, for a flat plate perpendicular to the walls, the resistance decreases as the channel widens, a result already found by Birkhoff, Plesset, and Simmons [Quart. Appl. Math. **8**, 151–168 (1950); these Rev. **12**, 297].

J. B. Serrin (Cambridge, Mass.).

Kufarev, P. P. On free-streamline flow about an arc of a circle. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 589–598 (1952). (Russian)

L'A. présente une nouvelle solution du problème déterminé du sillage plan, symétrique, posé en fluide indéfini relativement à un arc de cercle. La méthode est fondée implicitement sur l'analyticité de la correspondance $z=z(t)$, $t=\sqrt{f}$ (ou f est le potentiel complexe) le long de l'image de l'obstacle. La question est ramenée à une équation intégro-différentielle singulière que l'A. résout par approximations successives. Le processus est convergent et la solution unique pour de petites valeurs de l'angle au centre du profil. Il est à noter—et c'est un point important—que la méthode fournit des solutions approchées qu'il serait intéressant de comparer avec celles que donnent les procédés habituels.

Il semble que le présent mémoire apporte une idée nouvelle —sans doute, de faible portée—qui mérite d'être étudiée en détail. *J. Kravtchenko* (Grenoble).

Ehrich, Fredric F. Penetration and deflection of jets oblique to a general stream. J. Aeronaut. Sci. **20**, 99–104 (1953).

L'entrée d'un jet oblique dans un écoulement uniforme est étudiée à l'aide de la théorie des mouvements plans à potentiel. Le jet, issu d'un conduit à parois rectilignes, est supposé être séparé de l'écoulement principal par une ligne de courant et la vitesse amont du jet est choisie de telle manière qu'il n'y ait pas de discontinuité le long de cette ligne de séparation. En aval du jet il est supposé exister une zone de fluide au repos, limitée par une ligne de discontinuité sur laquelle la vitesse du jet est constante. Le problème est mis en équations au moyen de la méthode de l'hodographe. Les paramètres inconnus du mouvement et la forme des lignes de jet sont alors données au moyen d'intégrales.

Les formules résolutives sont explicitées dans les deux cas particuliers des jets à parois parallèles, perpendiculaires à l'écoulement principal et des jets à parois parallèles à cet écoulement. Le schéma dans lequel la zone de fluide immobile serait supprimée est aussi envisagé, et les résultats sont comparés avec ceux du schéma précédent. Enfin la solution de ce problème plan est comparée avec les résultats expérimentaux connus pour les jets issus d'un orifice circulaire et il est remarqué que le schéma plan donne une bonne approximation de ceux-ci. *R. Gerber* (Toulon).

Karal, F. C. The motion of a sphere moving parallel to a plane boundary. J. Appl. Phys. **24**, 147–151 (1953).

The problem of steady potential flow past a sphere located at an arbitrary distance from an infinite plane parallel to the free-stream direction is considered. It is shown that by introducing a bipolar coordinate system the boundary-value problem can be solved exactly for arbitrary ratio of the radius of the sphere and its distance from the infinite plane. The pressure distribution in the plane boundary for different geometric conditions is presented. It exhibits a minimum just below the sphere, maxima farther fore and aft, and gradual approach to free-stream pressure at infinity.
Y. H. Kuo (Ithaca, N. Y.).

Nadile, Antonio. Su alcune proprietà dello strato vorticoso non omogeneo. Univ. e Politecnico Torino. Rend. Sem. Mat. **11**, 279–287 (1952).

The author works out the equivalence of the velocity field induced by a vortex sheet, in general non-uniform and curved, with that induced by various other singularities. Most of the results follow by verbal changes from propositions already more or less familiar in the theory of magnetism. *C. Truesdell* (Bloomington, Ind.).

Liu, Hsien Chih. Über Schwingungen einer idealen Flüssigkeit in ellipsen- und kreisbogenförmigen, kommunizierenden Röhren. Ing.-Arch. **20**, 309–314 (1952).

The author considers vibrations under gravity of a non-viscous, incompressible fluid in a U-tube. He obtains the period in terms of a definite integral of an expression involving an infinite series when the tube is elliptical with constant cross section; in terms of a definite integral when the tube is circular with constant cross-section in the neighborhood of the free surfaces; and in terms of an infinite series when the tube is circular and of constant cross section throughout. The results are compared with the vibrations of a pendulum. *E. Pinney* (Berkeley, Calif.).

Liu, Hsien-Chih. Eigenschwingungen idealer Flüssigkeiten in Rohrleitungen mit verschiedenen Endquerschnitten. Ing.-Arch. **20**, 302–308 (1952).

A tube of variable cross section is closed at the bottom end by a piston-spring system and is filled with a non-viscous, incompressible fluid. The author analyses the vibrations of this system, neglecting lateral motion. When the cross section is uniform in the neighborhood of the upper surface, the equation of motion of the piston can be integrated completely in terms of elliptic integrals of the second kind. *E. Pinney* (Berkeley, Calif.).

Kawasaki, Tosio. A potential theory of arbitrary aerofoils. J. Jap. Soc. Appl. Mech. **5**, 1–3 (1952). (Japanese. English summary)

The author presents a method of successive approximation, by slightly modifying S. Moriguti's [J. Soc. Aeronaut. Sci. Nippon **5**, 223–230 (1938)], for determining a two-dimensional potential flow past a given thin airfoil section. Accuracy of the approximation is illustrated by a Joukowski wing. *Y. Komatu* (Tokyo).

Kawasaki, Tosio. A potential theory of lattices of arbitrary aerofoils. J. Jap. Soc. Appl. Mech. **5**, 3–7 (1952). (Japanese. English summary)

The author applies the same method as in his paper reviewed above to a given lattice of thin airfoil sections. A numerical illustration is given by a Joukowski wing lattice.
Y. Komatu (Tokyo).

Kuzmak, G. E. On a representation of the solution of the basic integro-differential equation of a wing. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 715–718 (1952). (Russian)

En utilisant les critères de N. Bary [C. R. (Doklady) Acad. Sci. URSS (N.S.) **37**, 83–87 (1942); ces Rev. **4**, 272] l'A. prouve que le système de fonctions:

$$f_n(\theta)=\left(1+\frac{\sin n\theta}{\pi cnb(\theta)}\right)\sin n\theta,\quad n=1, 2, \cdots, \infty,$$

[où c est une constante positive, $b(\theta)$ est une fonction donnée] est complet chaque fois que pour tout n on a:

$$\int_0^\pi [f_n(\theta)]^2 d\theta < \frac{4}{\pi}\left[\int_0^\pi f_n(\theta)\sin n\theta d\theta\right]^2.$$

Ce résultat est utilisé pour former le développement suivant les $f_n(\theta)$ de la solution de l'équation intégro-différentielle de l'aile. *J. Kravtchenko* (Grenoble).

Dolidze, D. E. On the limit of the solution of an unsteady linear boundary problem of hydrodynamics. Soobščeniya Akad. Nauk Gruzin. SSR **10**, 77–84 (1949). (Russian)

Poursuivant la série de ses recherches consacrées aux équations de Navier, l'A. étudie l'allure pour $t\to\infty$ du mouvement non-permanent d'un liquide visqueux, soustrait à l'action des forces extérieures, emplissant le domaine infini à trois dimensions D, extérieur à une surface assez régulière F_1; les équations du problème sont linéarisées et le vecteur vitesse $\mathbf{V}(P, t)$ au point $P \in D$ est assujetti aux conditions suivantes:

$$\mathbf{V}(P, 0)=0;\quad \mathbf{V}(N, t)=\mathbf{f}(N, t);\quad \lim_{P\to\infty}\mathbf{V}(P, t)=\mathbf{V}_0(t)$$

où N est le point courant sur F_1, $\mathbf{f}$ et $\mathbf{V}_0$ des fonctions vectorielles données. Moyennant des hypothèses de régularité convenables, le problème ci-dessus a une solution et une seule pour tout t fini. L'A. cherche à effectuer le passage à la limite $t\to\infty$ en admettant que les données ont des limites assez régulières pour $t=\infty$. On a alors les résultats suivants: (1) L'existence de $\lim_{t\to\infty}\mathbf{V}(P, t)$ n'est possible que si le problème stationnaire correspondant admet une solution. (2) Mais alors, les termes rectangulaires de $d\mathbf{V}(P, \infty)/dt$ sont, à grande distance de F_1, d'un ordre de grandeur supérieur aux termes linéaires conservés.
J. Kravtchenko (Grenoble).

Avališvili, L. I. Fundamental solutions of the linearized equations of unsteady motion of a viscous fluid. Soobščeniya Akad. Nauk Gruzin. SSR **12**, 397–400 (1951). (Russian)

L'A. construit le système de solutions fondamentales

$$v_{ik}(x_i, \xi_i, t),\quad p(x_i, t)\quad (i, j, k=1, 2, 3)$$

du système de Stokes-Oseen:

$$\nu\Delta\mathbf{v}-U(t)\frac{\partial\mathbf{v}}{\partial x_1}-\frac{\partial\mathbf{v}}{\partial t}=\frac{1}{\rho}\operatorname{grad} p, \tag{1}$$

$$\operatorname{div}\mathbf{v}=0$$

où $U(t)$ est une fonction donnée. Les v_{jk} et p_j vérifient pour chaque j le système (1) et satisfont, en outre, aux conditions:

$$\lim_{t=0,\,r>0} v_{jk}=\frac{1}{4\pi}\frac{\partial^2}{\partial x_j\partial x_k}\left(\frac{1}{r}\right);\quad \lim_{r=\infty} v_{jk}=0,$$

$$\int_0^t v_{jk}(x_i,\xi_i,t-\tau)d\tau=o\left(\frac{1}{r}\right)+\varphi_{jk}(x_i,\xi_i,t),$$

où $r=(\sum_1^3(x_i-\xi_i)^2)^{1/2}$ et où les φ_{jk} sont des fonctions continues. *J. Kravtchenko* (Grenoble).

Siezkin, N. A. Two problems on cutting into a viscous medium. Moskov. Gos. Univ. Učenye Zapiski **152**, Mehanika **3**, 56–60 (1951). (Russian)

L'A. donne les solutions approchées de deux problèmes plans, non-permanents, intéressant un milieu visqueux: (1) Le milieu emplit le domaine $y\gtrless 0$, $x\leqq 0$; une plaque rectangulaire, animée d'un mouvement de translation uniforme, parallèle à Ox pénètre dans le milieu. (2) Un coin solide de forme donnée, animé d'un mouvement de translation uniforme se déplace dans un milieu visqueux indéfini. Dans chaque cas, l'A. détermine le champ des vitesses et la résistance à l'avancement du solide. *J. Kravtchenko.*

Millsaps, Knox, and Pohlhausen, Karl. Thermal distributions in Jeffery-Hamel flows between nonparallel plane walls. J. Aeronaut. Sci. **20**, 187–196 (1953).

Les auteurs donnent la solution exacte du problème de la distribution des températures dans le mouvement permanent d'un fluide visqueux, incompressible, limité par deux parois planes, non parallèles, et maintenues à température constante.

La solution de Jeffery-Hamel du problème hydrodynamique permet de déterminer les profils des vitesses à l'aide de fonctions elliptiques. Il est montré que dans ce cas l'équation de l'énergie, qui fournit la distribution des températures, peut être réduite à une équation différentielle linéaire à coefficients variables. Les solutions numériques sont données dans le cas des écoulements divergents et convergents, avec un angle d'ouverture total de 10°, pour trois valeurs du nombre de Reynolds et cinq valeurs du nombre de Prandtl. *R. Gerber* (Toulon).

Geršuni, G. Z. On free heat convection in the space between vertical coaxial cylinders. Doklady Akad. Nauk SSSR (N.S.) **86**, 697–698 (1952). (Russian)

Assuming laminar flow of an incompressible fluid, a closed solution for the velocity and temperature distribution is obtained for the case of axial symmetry and the boundary condition of zero flow velocity along the vertical walls. The temperature is given by $T=1-\ln r/\ln\rho$ and the velocity by:

$$V=\tfrac{1}{4}\mathrm{Pr}\cdot\mathrm{Gr}\left[B_1(r^2-1)+\frac{\ln r}{\ln\rho}(r^2+B_2)\right]$$

where r is the ratio of radial position to inner radius, ρ is the ratio of outer to inner radius, and B_1 and B_2 are functions of ρ. *N. A. Hall* (Minneapolis, Minn.).

Šestopalov, V. P. On a particular solution for a heated boundary layer in a diffusor. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 613–616 (1952). (Russian)

Consideration is given to the laminar, incompressible, thermal boundary layer developed near the leading edge of a wedge. Applying the known solution for the velocity boundary layer, the thermal equation including the effects of viscous dissipation is reduced by suitable change of independent variable to the ordinary hypergeometric differential equation. The formal solution of this in the general case considered is set forth. As the hypergeometric function parameters depend algebraically on the thermal diffusivity of the fluid, a treatment of a numerical solution by use of a few terms in the series is considered for a special application and shown to be satisfactory. *N. A. Hall.*

Emmons, H. W. The laminar-turbulent transition in a boundary layer. I. J. Aeronaut. Sci. **18**, 490–498 (1951).

The author gives a "probability transition theory" to explain the phenomena observed when a laminar boundary layer becomes turbulent. Each point in the transition region is regarded as fully turbulent part of the time. The probability that a region becomes turbulent and the manner in which the turbulent region spreads are regarded as the main factors governing the phenomena of transition. With reasonable assumptions regarding these factors, the author is able to obtain the dependence of the skin coefficient on Reynolds number in general agreement with experiments. *C. C. Lin* (Cambridge, Mass.).

Thomas, L. H. The stability of plane Poiseuille flow. Physical Rev. (2) **86**, 812–813 (1952).

The author solves the classical problem of the stability of plane Poiseuille flow by direct numerical integration of the Sommerfeld equation. Definite instability is found beyond a Reynolds number of 5800. Both the nature of the neutral curve and its location are found to be in close agreement with those obtained by this reviewer by asymptotic methods [Quart. Appl. Math. **3**, 277–301 (1946); these Rev. **7**, 346]. *C. C. Lin* (Cambridge, Mass.).

Lessen, Martin. On the stability of plane parallel laminar flows to two- and three-dimensional disturbances. J. Aeronaut. Sci. **19**, 431–432 (1952).

The author discusses the role of the energy equation in connection with the reduction of the stability of a three-dimensional disturbance to a similar two-dimensional problem. *C. C. Lin* (Cambridge, Mass.).

Laurmann, J. A. Stability of the compressible laminar boundary layer with an external pressure gradient. Coll. Aeronaut. Cranfield. Rep. no. **48**, 3+64 pp. (12 plates) (1951).

The theory of stability of boundary layer in a compressible fluid, as developed by Lees and Lin [NACA Tech. Note no. 1115 (1946); these Rev. **8**, 236], is applied to the case of flow with pressure gradient. Analysis is given to justify the application. Calculation of the neutral stability characteristics gave the following general results. (i) For any given free stream Mach number, an increasing negative pressure gradient increases the stability of the laminar layer, i.e., the value of the stability limit (defined as the Reynolds number below which there is laminar stability) increases, and the range of unstable disturbance frequencies and velocities becomes smaller. (ii) With a positive pressure

gradient the stability decreases as the Mach number increases. (iii) For a sufficiently large negative pressure gradient there is a range of Mach number, varying from about 1.3 to 2.4, for which the boundary layer is completely stable for all Reynolds numbers; however, the stability limit always decreases at large enough values of the Mach number. *C. C. Lin* (Cambridge, Mass.).

Batchelor, G. K. The effect of homogeneous turbulence on material lines and surfaces. Proc. Roy. Soc. London. Ser. A. **213**, 349–366 (1952).

The author studies the rate of extension of material lines and surfaces in homogeneous turbulence. In particular, he considers the apparent difficulty that the volume of an elementary parallelepiped can remain unchanged while each side is being stretched. This is analyzed in terms of change of the angles between the sides. The author then considers the effect of these kinematic results on the convection of scalar quantities such as the density of smoke in air and vector quantities such as the vorticity of the fluid.

C. C. Lin (Cambridge, Mass.).

Ward, G. N. On the integration of some vector differential equations. II. Application to the linearized theory of steady compressible fluid flow. Quart. J. Mech. Appl. Math. **5**, 441–446 (1952).

Applications are given of the general formulae developed in part I [same J. **5**, 432–440 (1952); these Rev. **14**, 651]. Expressions are calculated for the velocities due to arbitrary source and vortex distributions according to the linearised theory of compressible flow. It is pointed out (rightly so) that a formula given by the present reviewer for the discontinuity of the normal velocity component across a surface distribution of sources in supersonic flow [Quart. J. Mech. Appl. Math. **1**, 408–432 (1948); these Rev. **10**, 410], equation (46)] is incorrect. One may add that the reason for this is an algebraic mistake in the derivation of (46) from (44). Barring such mistakes, Ward's formula (60) can also be derived immediately from equation (44) in the reviewer's paper. *A. Robinson.*

Saito, Osamu. On some exact solutions of the two-dimensional steady flow of compressible fluid. J. Phys. Soc. Japan **6**, 243–254 (1951).

The author writes down the equations satisfied by the potential φ and velocity angle θ as functions of the speed q and stream-function ψ in a plane steady gas flow. [Cf. Saito and Amemiya, same J. **5**, 201–202 (1950); these Rev. **13**, 83]. He then discusses two particular solutions of these equations. *L. Bers* (New York, N. Y.).

Bunimovič, A. I. On the discharge of a gas with high subsonic velocities. Moskov. Gos. Univ. Učenye Zapiski **152**, Mehanika **3**, 117–186 (1951). (Russian)

On doit à Tchaplyguine une méthode, aujourd'hui classique, de résolution du problème plan symétrique du jet gazeux, s'échappant d'un vase à parois polygonales données et dont le nombre de Mach $M<0.5$ environ. L'A. cherche à étendre les calculs de Tchaplyguine au cas où $1-M>0$ est petit et où les parois fixes symétriques sont curvilignes.

Au départ, l'A. fait une hypothèse simplificatrice quelque peu surprenante. Soit $K=(1-(2\beta+1)\tau)/(1-\tau)^{2\beta+1}$, où $(\beta+1)/\beta$ est le rapport local des chaleurs spécifiques et τ est le rapport des carrés des modules des vecteurs vitesses au point courant et au point de référence. Bien que K varie rapidement en fonction de M dans le voisinage de $M=1$, l'A. admet qu'en première approximation, on peut prendre $K=K_0$ (K_0 étant une constante convenable) pour les écoulements qu'il étudie. Ceci fait, une transformation, due à Leibenson, permet d'exprimer les solutions du système de Tchaplyguine ainsi que toutes les caractéristiques du mouvement au moyen d'une fonction analytique ω de variable complexe. Les équations finales en ω deviennent analogues à celles du problème classique du sillage plan symétrique, la fonction ω jouant le rôle de l'expression classique de Levi-Cività. Au total, on peut former une infinité de solutions indéterminées du problème des jets gazeux à partir d'une fonction ω choisie à priori; en écrivant que les parois ainsi obtenues coincident avec les parois données, on obtient une équation intégro-différentielle (E) du problème déterminé, analogue à l'équation classique de Villat. L'A. tente de résoudre (E) par une méthode des approximations successives, dont le principe est dû à Nekrasov et qui n'échappe pas aux critiques bien connues, présentées par Weinstein.

Il serait intéressant d'étudier la validité physique des hypothèses de l'A. Les nombreux calculs numériques qu'on trouvera dans son travail faciliteraient la comparaison avec l'expérience de ses prévisions théoriques.

J. Kravtchenko (Grenoble).

Kopzon, G. I. Two-dimensional impact on a slightly compressible ideal fluid. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 719–722 (1952). (Russian)

Un problème plan de choc intéressant un fluide parfait, peu compressible, est ramené, en première approximation, au problème aux limites suivantes. Dans la bande: $0\leqq y\leqq l$, $-\infty\leqq x\leqq\infty$, déterminer la solution $\varphi(x, y, t)$ de l'équation $a_0^2\Delta\varphi-\partial^2\varphi/\partial t^2=0$, telle que (1) $\partial\varphi/\partial x|_{t=0}=U$; (2) $\partial\varphi/\partial y|_{y=0}=0$ pour $0\leqq t\leqq\infty$, $|x|>c$; (3) $\partial\varphi/\partial y|_{y=l}=0$ pour $0\leqq t\leqq\infty$, $-\infty\leqq x\leqq\infty$; (4) $\partial\varphi/\partial y|_{y=0}=f(x, t)$ pour $t>0$, $|x|<c$; a_0, U, l sont des constantes données et $f(x, t)$ est une fonction donnée. C'est ce problème que l'A. résout par une superposition d'un nombre fini de potentiels calqués sur ceux, utilisés par Falkovitch [cf. même journal **11**, 171–176 (1947); ces Rev. **9**, 543]. Un théorème d'unicité est énoncé; de même l'A. énonce des propriétés de la solution ($\partial^2\varphi/\partial t^2|_{t=0}=\infty$; discontinuité de $\partial^2\varphi/\partial y^2$). Les résultats généraux ci-dessus sont explicités dans le cas d'une source quasi-ponctuelle de perturbation. *J. Kravtchenko.*

*Ghaffari, A. G. **The hodograph method in gas dynamics.** University of Tehran. Faculty of Science. Publ. no. 85, 1950. ii+iv+129 pp.

This is a report on some aspects of the hodograph method. The fundamental gas dynamical equations are derived and discussed in Chapter I. Chapter II contains the linearization of the equations by the Legendre transformation and the Molenbroek-Chaplygin transformation. Exact and approximate solutions of Chaplygin's equation, their asymptotic properties, and velocity-correction factors are discussed in Chapters III–V. An introduction to Bergman's method is given in VI. Chapter VII, based on the author's thesis, discusses approximate solutions of the hodograph equation in the transonic range. The author's method seems to amount to the use of Tricomi's equation as an approximate hodograph equation. In the last chapter simple waves (supersonic solutions with one-dimensional hodograph) are discussed. These results are described in Ghaffari, Proc. Internat. Congress Math., Cambridge, Mass., 1950, Amer. Math. Soc., Providence, R. I., 1952, vol. I, p. 630.

L. Bers (New York, N. Y.).

Tierney, John A. An approximation to transonic flow of a polytropic gas. Amer. J. Math. **75**, 43–56 (1953).

The equation of state of a gas whose flow is governed by Frankl's equation, $\psi_{\sigma\sigma}+K(\sigma)\psi_{\theta\theta}=0$, is determined by the choice of the function

$$K(\sigma)=\sigma-b\sigma^3+\cdots;\quad b=\tfrac{1}{2}(2\gamma+5)(\gamma+1)^{-1/3},$$

where γ is the adiabatic exponent (case of a polytropic gas). The case $K(\sigma)=\sigma$ is called the Tricomi gas or a T_1-gas. The case $K(\sigma)=\sigma-b\sigma^3$ is called a T_3-gas. The variable σ has been introduced by Frankl [Izvestiya Akad. Nauk SSSR. Ser. Mat. **9**, 121–143 (1945)=NACA Tech. Memo. no. 1155 (1947); these Rev. **7**, 496; **8**, 541] as the integral

$$\sigma=\int_{p^*}^{p}q^{-2}dp=\sigma(p),$$

where p^*, the sonic pressure, is defined by $q_{pp}(p^*)=0$ ($q=q(p,\psi)$=Bernoulli function, p=pressure, ψ=stream function). The sonic speed is given by $q^*=q(p^*)$, the sonic density is denoted by ρ^*. In particular, $\sigma=0$ yields the sonic values q^*, p^*, ρ^*. If two functions $K(\sigma)$ have the same expansions in positive integral powers of σ up to and including the term in σ^n, the graphs in the (ρ, p)-plane of the corresponding equations of state have contact of order at least $n+1$ at the sonic point (ρ^*, p^*). For example, the equations $\psi_{\sigma\sigma}+(\sigma-b\sigma^3)\psi_{\theta\theta}=0$ can be made to have contact of the third order at the sonic point (ρ^*, p^*). In the concluding section the author studies a particular example of transonic flow for a T_3-gas by mapping from the (σ, ψ)-plane to the physical z-plane. Setting $\sigma=0$ the author obtains parametric equations for the sonic line with ψ serving as parameter. In an appendix the graphs of $K=K(\sigma)$ for a T_1-gas and a T_3-gas are compared with the graph for a polytropic gas.

M. Pinl (Dacca).

Kaplan, Carl. On a solution of the nonlinear differential equation for transonic flow past a wave-shaped wall. NACA Rep. no. **1069**, ii+11 pp. (1952).

Supersedes NACA Tech. Note no. **2383** (1951); these Rev. **12**, 875.

***Thomas, T. Y. A theory of the stability of shock waves.** Proceedings of the Midwestern Conference on Fluid Dynamics, 1950, pp. 109–120. J. W. Edwards, Ann Arbor, Michigan, 1951.

This article gives a summary of results of the local theory of stability of stationary shocks presented in a number of previous papers of the author [these Rev. **8**, 611; **10**, 271, 494, 758; **11**, 479]. The problem of the stability of shock waves in the large is suggested for serious study.

C. C. Lin (Cambridge, Mass.).

Giese, J. H., and Cohn, H. Two new non-linearized conical flows. Quart. Appl. Math. **11**, 101–108 (1953).

The authors have discovered two Prandtl-Meyer types of flow in a nonlinear conical field, which can be interpreted as simple waves in a plane normal to the undisturbed flow. If the body consists of plane boundaries belonging to one member of the family, the boundary conditions are obviously satisfied. One example given is an airplane with a swept-forward wing of positive dihedral and with a fin. The shock attached at the leading edge lies under the wing and above it; there are two families of simple waves. To allow slightly curved wings, the possibility of interaction is considered. The process is indicated but no concrete result is given.

Y. H. Kuo (Ithaca, N. Y.).

Robinson, A. Aerofoil theory of a flat delta wing at supersonic speeds. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2548** (**10,222**), 21 pp. (1952).

The pressure distribution on a delta wing with leading edges inside the Mach cone is determined by separating the potential equation in the author's hyperboloido-conal coordinates. The results were published first in 1946 [Ministry of Supply [London], Aeronaut. Res. Council Rep. no. **9954** (1946); Roy. Aeronaut. Establishment Rep. Aeronaut. no. **2151** (1947)] or about the same time as the solutions given by C. E. Brown [NACA Rep. no. **839** (1946); these Rev. **10**, 339] and H. J. Stewart [Quart. Appl. Math. **4**, 246–254 (1946); these Rev. **8**, 109]. While the author's method appears rather more involved than Stewart's application of conical flow theory, it covers a wider range of possible wing motions, as evidenced by later applications [Robinson and Hunter-Tod, Coll. Aeronaut. Cranfield Rep. no. **10** (1947); these Rev. **9**, 479; Robinson, ibid. no. **16** (1948); these Rev. **10**, 163; J. Roy. Aeronaut. Soc. **52**, 735–752 (1948); S. V. Falkovich and M. D. Haskind, Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 371–376 (1947); these Rev. **9**, 114; P. Germain and R. Bader, Recherche Aéronautique **1949**, no. 11, 3–13 (1949); these Rev. **11**, 476].

J. W. Miles.

Charnes, Abraham. Wing-body interaction in linear supersonic flow. Abstract of a thesis, University of Illinois, 1947, Urbana, Ill., 1952. 3+i pp.

Paničkin, I. A. Supersonic flow of a gas about an oscillating wing profile. Moskov. Gos. Univ. Učenye Zapiski **152**, Mehanika **3**, 226–253 (1951). (Russian)

Operational calculus has been applied to the linearized two-dimensional flow about an airfoil performing small harmonic oscillations in a uniform supersonic stream to simplify derivations of the lift previously given by the author [Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 165–170 (1947)=Grad. Div. Appl. Math., Brown Univ. Translation **A9-T-21** (1948); these Rev. **9**, 545,632] and by Krasil'ščikova [Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 147–164 (1947); these Rev. **9**, 392].

J. H. Giese.

Barenblatt, G. I. On self-similar motions of a compressible fluid in a porous medium. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 679–698 (1952). (Russian)

The density ρ of a gas is assumed to satisfy

$$c\frac{\partial\rho}{\partial t}+x^{-s}\frac{\partial}{\partial x}\left[x^s\left(-\frac{\partial p^k}{\partial x}\right)^m\right]=0,$$

a special one-dimensional form of Leibenson's equation for turbulent filtration where it is required that $\partial\rho/\partial x\leqq 0$ [L. S. Leibenson, Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **9**, 7–10 (1945); these Rev. **7**, 95]. The index $s=0$ for the plane wave case, $s=1$ when there is cylindrical symmetry, and $s=2$ for the spherically symmetric case. The filtering medium determines the constant c, m characterizes the turbulence of the flow ($0.5\leqq m\leqq 1$, with $m=1$ for laminar flow), and $k-1=n$ is the index of polytropy of the gas. With the conditions $\rho(x,+0)=0$ for $x>0$, and

$$\lim_{x\to 0}x^s\left(-\frac{\partial\rho^k}{\partial x}\right)^m=\tau t^q,$$

the author finds the form of the solutions $\rho(x)$. He shows essentially that when $km>1$, the velocity of propagation of the gas front is finite, i.e., the curve $\rho(x)$ meets the x-axis.

But when $km<1$, the gas front is propagated with infinite velocity. *R. E. Gaskell* (Seattle, Wash.).

Westervelt, Peter J. The theory of steady rotational flow generated by a sound field. J. Acoust. Soc. Amer. **25**, 60–67 (1953).

The author claims to derive from the Navier-Stokes equations a relation governing the time average of the vorticity. This he interprets in terms of a perturbation procedure. He treats especially the cases when the relations of the linearized equations are irrotational or solenoidal. Hence he claims to derive results in one case which agree with those of Eckart [Physical Rev. (2) **73**, 68–76 (1948)]; in another, results which correct an earlier treatment by Rayleigh. The reviewer finds the use of unstated assumptions and undefined terms, as well as numerous misprints, some obstacle in following even the purely formal manipulations which the author uses to infer his results. *C. Truesdell.*

Mukherjee, Santi Ram. Motion of a compressible fluid with varying viscosity given by $\mu=\mu_0+\epsilon_1 x$ for positive values of x, with reference to sound waves. Proc. Nat. Acad. Sci. India. Sect. A. **15**, 91–99 (1946).

The author constructs some approximate solutions exponential in time for the problem stated in the title. There is, however, no "reference to sound waves" or indication of any other application in the paper. *C. Truesdell.*

Isakovič, M. A. Scattering of waves from a statistically rough surface. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **23**, 305–314 (1952). (Russian)

The author considers mainly sound waves which are perfectly reflected by a rigid surface; that the statistical irregularities are large compared with the sound wavelength is stated to distinguish the paper from previous work, apart from M. L. Antokol'skiĭ [Doklady Akad. Nauk SSSR (N.S.) **62**, 203–206 (1948)]. Starting from the Kirchhoff formula for the potential and making simplifications, he calculates the mean potential and mean-square potential in terms of distribution functions for random variables defining the irregularities. Details are given in the case where certain parameters are normally distributed. Directivity charts are drawn for particular cases. Finally it is indicated how to extend the method to electromagnetic waves with an ideally conducting surface.

F. V. Atkinson (Ibadan).

Elasticity, Plasticity

Tonolo, Angelo. Sopra un problema di Darboux della meccanica dei mezzi continui. Ann. Univ. Ferrara. Sez. VII. (N.S.) **1**, 103–109 (1952).

The problem to which the title refers is to characterize those finite deformations in which there exists at each initial point at least one triply orthogonal set of directions which is carried into a parallel triply orthogonal set, and to characterize the subclass in which these are tangent to the curves of intersection of a triply orthogonal family of surfaces in one or the other condition. The author obtains Darboux's results by tensorial methods, employing general coordinates.

C. Truesdell (Bloomington, Ind.).

Nardini, Renato. Sul valor medio dello stress per particolari sollecitazioni. Ann. Univ. Ferrara. Sez. VII. (N.S.) **1**, 89–91 (1952).

The author points out some special cases of Signorini's theorem of stress averages [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **15**, 151–156 (1932), and later papers] in which the result is especially simple.

C. Truesdell (Bloomington, Ind.).

Conway, H. D. Some problems of orthotropic plane stress. J. Appl. Mech. **20**, 72–76 (1953).

The Airy stress function in the form of Fourier series or Fourier integrals of harmonic or biharmonic functions has been extensively used for plane stress problems in isotropic elasticity relating to a half-plane or a wedge. The author adapts these solutions to the corresponding problems when the material is orthotropic, and derives the stress functions, stresses and displacements for the following problems: (a) half-plane subjected to a normal loading; (b) wedge subjected to axial loading; (c) half-plane subjected to a normal force applied through a rigid punch; (d) half-plane subjected to tangential loading; (e) wedge subjected to transverse loading; (f) force applied at a point in an infinite plane; (g) analysis of deep beams. In (g) a doubly infinite system of equations for the constants involved require to be solved, and it is pointed out that this can be done by considering only a finite number of terms or by iteration.

R. M. Morris (Cardiff).

Bölcskei, E. Déformation des voiles minces. Acta Tech. Acad. Sci. Hungar. **5**, 489–506 (1952). (Russian summary)

In the extensional or "membrane" theory of shells, the stress resultants can often be determined independently of the displacements. The latter then satisfy a certain system of partial differential equations. The author notices this well-known and obvious fact. The specific form of all equations in a more general theory was obtained in intrinsic coordinates by Love in his original paper [see, e.g., Elasticity, 4th ed., Cambridge, 1927, §329], and several authors have given tensorial treatments, valid in all coordinate systems. The author at some length derives the appropriate equations in rectangular Cartesian coordinates. *C. Truesdell.*

Zerna, W. Zur Berechnung der Randstörungen kreiszylindrischer Tonnenschalen. Ing.-Arch. **20**, 357–362 (1952).

Green and the author [Quart. J. Mech. Appl. Math. **3**, 9–22 (1950); these Rev. **11**, 701; cf. also Zerna, Z. Angew. Math. Mech. **30**, 370–374 (1950); these Rev. **12**, 770] have proposed a simplified theory of the bending of thin shells. One of the principal approximations consists in essence in taking the shear resultants and the resultant twisting moments as symmetric, while in Love's theory they are not. In this paper the author considers the simplifications which result when the shell is a circular cylinder. He is then able to reduce the entire problem to that of finding an appropriate solution of a simple linear partial differential of fourth order with constant coefficients. This solution he obtains in the form of a Fourier series. The harmonics of nth order occur in pairs, each of which contains sixteen arbitrary constants. The remaining parameters are expressed in terms of n and the various constants of the problem by explicit formulae, which are also presented graphically. The author writes the explicit expressions for the corresponding harmonics for the eight static resultants and the three displace-

ments. With these formulae the problem may be considered solved, within the proposed simplified theory. The results, while elaborate, are much simpler than those obtainable by Love's theory (which also, it must be remembered, is open to some question). It is to be hoped that in a later work the author will give examples of special solutions and compare them with their counterparts according to Love's theory.

C. Truesdell (Bloomington, Ind.).

Ban, Shizuo. Formänderung der hyperbolischen Paraboloidschale. Mem. Fac. Eng. Kyoto Univ. **14**, 181–194 (1952).

After considering the membrane stresses and strains, the differential equation is obtained for the displacements of an arbitrary cylindrical shell. The solution of this equation is obtained for the special case of a hyperbolic paraboloidal shell and a few types of loading are discussed. For the case of a shell roof consisting of hyperbolic paraboloidal shells, it is shown how bending stresses may be eliminated.

H. D. Conway (Ithaca, N. Y.).

Karunes, B. A note on the problem of dislocation in a semi-infinite plate containing a circular hole. Indian J. Phys. **26**, 442–444 (1952).

This is a short note in which the author writes down suitable solutions for the stress function in bipolar coordinates for the case of an unstressed circular hole near the straight edge of a semi-infinite plate. The particular solutions he chooses are shown to give multiple-valued terms in the displacements corresponding to certain types of dislocation, viz., a wedge-shaped fissure, and a parallel fissure. Other terms are also added to the stress function to ensure that there are no stresses over the boundary. From this stress function the circumferential stresses over the boundary of the hole and the straight edge are derived.

R. M. Morris (Cardiff).

Ilkova, A. G. Bending of a circular plate lying on an elastic half-space under the action of an axially symmetrical uniformly distributed load. Moskov. Gos. Univ. Učenye Zapiski **152**, Mehanika **3**, 202–225 (1951). (Russian)

The fourth order differential equation for the deflection of the plate with free edges described in the title, which was given by Lagrange and S. Germain, and the integral equation for the sag of the foundation which was given by Boussinesq and Schleicher, together with the boundary conditions contain the solution of the problem. The unknown reaction of the foundation present in both equations makes it very difficult to find the exact solution. The problem was tackled before by Borovik and Gorbunov-Posadov (references not given). They assumed the reaction to be a power series in radius vector r, and the problem was reduced to an infinite system of equations determining the coefficients of the series. The system was not solved, but the first few coefficients were approximately determined.

The author of this paper assumes also that the reaction is a power series in r but of a very special form. She also obtains the infinite system of equations in coefficients of the series and manages very ingeniously to solve it. The solutions, i.e., the coefficients, are infinite series themselves. The solutions are checked and the series representing the reaction is tested for convergence.

T. Leser.

Grioli, Giuseppe. Sul problema di de Saint-Venant nei solidi cristallini. Rend. Sem. Mat. Univ. Padova **21**, 228–242 (1952).

For an isotropic elastic body, the author constructs a partial formulation of St. Venant's problem of the deformation of prisms subject to terminal load as a stress problem. Hence he derives relations between the applied loads and certain stress averages. He then specializes the type of anisotropy by requiring that 10 of the elastic coefficients vanish. For such media, he is able to derive a rather complicated minimum theorem to replace Menabrea's theorem. In the author's integral to be minimized occurs the static resultant of the load, and the side conditions include properties of the average stresses. He proposes an iteration process to solve the equations so obtained, illustrating the results in the case of torsion and non-uniform bending.

C. Truesdell (Bloomington, Ind.).

Tsien, H. S. Similarity laws for stressing heated wings. J. Aeronaut. Sci. **20**, 1–11 (1953).

Based on Kirchhoff's bending hypothesis, differential equations satisfied by the stress function and the lateral deflection are derived for a heated plate, on the supposition that both the thermal strain and stress vary linearly with the temperature increment. In comparison with the case of an unheated plate, the differential equations would assume a similar form if the unheated plate were acted upon by body forces. The proposed problem is: for the same planform with one end built-in, is it possible to find a thickness distribution and loading for the unheated plate such that the solutions for heated and unheated plates are proportional? The conditions of similarity finally give a set of relations among Young's modulus distribution for the heated plate, the thickness distributions of the unheated and heated plates, and the similarity parameters. By properly choosing two of the four arbitrary parameters, the geometry of the unheated plate can be determined.

To specify the loads on the unheated plate, it is shown that the pressure loading can be expressed in terms of the measured plane strain distribution of the unheated plate and the given thermal bending moment and pressure loading on the heated plate. The body force is given by the gradient of the sectional tension at the boundary of the heated plate. With these questions settled, the stresses in the heated plate can be calculated. As a practical method an alternative test procedure is also given. At the end of the paper, a possible application to the case where the plate has a structure, such as a wing, is briefly discussed.

Y. H. Kuo.

Levy, Samuel. Influence coefficients of tapered cantilever beams computed on SEAC. J. Appl. Mech. **20**, 131–133 (1953).

Cattaneo, Carlo. Sulla torsione di due sfere elastiche a contatto. Ann. Scuola Norm. Super. Pisa (3) **6**, 1–16 (1952).

The author states that the problem stated in the title was treated by J. L. Lubkin [J. Appl. Mech. **11**, 183–187 (1951); these Rev. **12**, 878] but that his own method of solution, which he outlines here, was obtained independently. This problem is formulated within the Hertzian theory of bodies in contact. The author first effects one integration, thus deriving a single integral equation. He discusses some approximate results and then outlines the formal steps in an exact solution in series. Rigorous justification and illustration of the results in special cases are reserved for a later work.

C. Truesdell.

Satô, Yasuo. Study on surface waves. VII. Travel-time of Love-waves. Bull. Earthquake Res. Inst. Tokyo **30**, 305–317 (1952). (Japanese summary)

For part VI see same Bull. **30**, 101–120 (1952); these Rev. **14**, 337.

Ashwell, D. G. A characteristic type of instability in the large deflexions of elastic plates. I. Curved rectangular plates bent about one axis. II. Flat square plates bent about all edges. Proc. Roy. Soc. London. Ser. A. **214**, 98–118 (1 plate) (1952).

In the first part of this paper the author utilizes Marguerre's equations for finite deflections of thin, shallow shells [Proc. Vth Internat. Congress Appl. Mech., Cambridge, Mass., 1938, Wiley, New York, 1939, pp. 93–101] for the analysis of pure bending of thin plates with initial transverse curvature. It is found, by means of an exact solution of the problem that such plates exhibit the type of instability characteristic of thin-walled structures which depend for their stiffness on curvatures. Curves are given for the relation between bending moment and curvature and an expression is obtained for the critical bending moment at which instability occurs. Briefly considered is the problem of the plate with initial longitudinal curvature, for which a significant correction is found to an earlier analysis by Lamb [Proc. London Math. Soc. **21**, 119–146 (1890)]. The second part of the paper deals with originally flat square plates loaded by bending moments applied to all four edges. It is found that instability may occur in the sense that instead of the expected continuation of the results of the linear theory, with membrane stresses due to finite deformation, we may have a jumping to a state of nearly cylindrical bending. *E. Reissner* (Cambridge, Mass.).

Duberg, John E., and Wilder, Thomas W., III. Inelastic column behavior. NACA Rep. no. **1072**, iii+16 pp. (1952).

Supersedes NACA Tech. Note no. **2267** (1951); these Rev. **12**, 881.

Onat, E. T., and Prager, W. Limit analysis of arches. J. Mech. Phys. Solids **1**, 77–89 (1953).

It is well known that axial forces in frames and other continuous structures lower the plastic collapse load. The present paper is the first systematic treatment of the effects of axial forces on both the kinematics and the statics of the collapse state, taking account of the fundamental theorems of limit analysis due to Drucker, Prager, and Greenberg. However, geometry changes under load are neglected so that questions of stability require separate investigation. In the circular-arch examples treated in the paper the axial forces did not appreciably affect the collapse mechanism, but the collapse load was lowered by about 13 percent (symmetrical loading) and by about 10 percent (unsymmetrical loading). *P. S. Symonds* (Providence, R. I.).

Geiringer, Hilda. Über die Charakteristiken des vollständigen ebenen Plastizitätsproblems. Z. Angew. Math. Mech. **32**, 379–387 (1952). (English, French and Russian summaries)

The paper is primarily concerned with problems of plane plastic flow; to problems of plane stress the results apply only at the onset of plastic flow in a sheet of uniform thickness. The plastic potential may differ from the yield function, but both are tacitly assumed to be of class C^1 in the stress components. Three families of curves are considered: the characteristics of the stress field, the characteristics of the velocity field, and the lines of maximum shearing stress. In the classical problem of plane plastic flow under the yield condition and flow rule of v. Mises, these three families are known to coincide. The author establishes the conditions under which two of these families coincide. For stress and velocity characteristics to coincide, the yield function must be used as the plastic potential. If the stress characteristics and the lines of maximum shearing stress are to coincide, the sum of the derivatives of the yield function with respect to the principal stresses must vanish. Finally, if the velocity characteristics are to coincide with the lines of maximum shearing stress, the sum of the derivatives of the plastic potential with respect to the principal stresses must vanish. The yield function defines a one-parameter family of Mohr circles, and there is a well-known relation between the slope of their envelope and the angle formed by the stress characteristics. The author establishes a similar relation for the angle formed by the velocity characteristics, using a one-parameter family of Mohr circles defined by the plastic potential. *W. Prager* (Providence, R. I.).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Schopper, Herwig. Zur Optik dünner doppelbrechender und dichroitischer Schichten. Z. Physik **132**, 146–170 (1952).

The author discusses the equations of multiple reflection of light in an anisotropic, diochroitic thin layer for a parallel beam entering at an angle. He investigates monoaxial layers as well as biaxial layers, provided the optical axis is normal to the planes. The formulae computed for finite layers are specialized for very thin layers. The author also investigated the results of his formulae for an incident beam, whose angle is larger than the angle of total reflection. A critical review of a system of formulae developed by K. Försterling with respect to their application to anisotropic layers and a method to determine the thickness of these layers are appended. *M. Herzberger* (Rochester, N. Y.).

v. Fragstein, C. Reziprozität der Durchlässigkeit einer beliebigen Mehrfachschicht für elektromagnetische Wellen. Optik **9**, 337–359 (1952).

The author derives from the electromagnetic theory the fact that a multiple layer consisting of many components has the same transparency in one direction as in the other if and only if the index of absorption is the same in object and image space. *M. Herzberger* (Rochester, N. Y.).

Zernov, N. V. On diffraction of plane-cylindrical electromagnetic waves. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **21**, 1066–1075 (1951). (Russian)

Let a semi-infinite cylindrical wave-guide be joined to the circular opening in an infinite conducting plane. The author presents a method for approximating to the field of monochromatic cylindrical waves propagated along the wave-guide. The cases of "magnetic" and "electric" waves are considered separately. The method leads to infinite sets

of equations, stated to have absolutely convergent determinant. A numerical case is worked in which the first three approximations, derived by abbreviating the equations, appear to converge rapidly. Points of contact with the integral-equation method and the Ritz method are noted; see also Kisun'ko [Doklady Akad. Nauk SSSR (N.S.) **66**, 863–866 (1949); these Rev. **10**, 764]. *F. V. Atkinson.*

Brehovskih, L. M. Diffraction of waves from an uneven surface. I. General theory. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **23**, 275–288 (1952). (Russian)

The author recapitulates in slightly more detail the material of his previous notes [Doklady Akad. Nauk SSSR (N.S.) **79**, 585–588 (1951); **81**, 1023–1026 (1951); these Rev. **13**, 599, 605] on the diffraction of monochromatic plane acoustic or electromagnetic waves from a doubly corrugated plane, it being here emphasized that the radius of curvature should be large at all points compared to the wave-length. Additional matter includes a survey of previous related work, a discussion of the conservation of energy for a perfectly reflecting surface, and criteria for the applicability of the assumptions of geometrical optics when the wave is actually or nearly tangential. The mathematical reasoning used is mainly heuristic in character. *F. V. Atkinson.*

Brehovskih, L. M. Diffraction of waves from an uneven surface. II. Applications of the general theory. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **23**, 289–304 (1952). (Russian)

The corrugated plane is now restricted to a strip. For the acoustic case the author takes from the foregoing paper the expression for the potential as a series of multiple integrals, from which by rough and ready methods approximate formulae are found intended to hold at large distances. Thirteen diagrams are given showing dependence of field strength on direction for various angles of incidence and proportions of sound wave-length, amplitude and wave-length of the sinusoidal surface. An independent method is sketched for normal incidence, and is said to confirm the previous calculations for amplitudes of waves corresponding to optical reflection. Finally diagrams are given and calculations sketched for cases when the surface is corrugated by superimposed waves or when its cross-section is a broken line. *F. V. Atkinson* (Ibadan).

Pryce, M. H. L. The diffraction of radio waves by the curvature of the Earth. Advances in Physics **2**, 67–95 (1953).

The problem is to obtain the electromagnetic field produced by an electric dipole above the surface of the sphere $R=a$. The field can be easily expanded into a series of spherical harmonics but in practical applications 10^3–10^5 terms are needed to give a reasonable approximation of the diffracted field. By means of complicated analysis, G. N. Watson [Proc. Roy. Soc. London. Ser. A. **95**, 83–99 (1918)] showed that the series can be written as a contour integral which can be expanded into a series of residues. The sum of a few terms of this residue series gives an excellent approximation to the diffracted field. In this paper the author uses from the beginning the fact that in practical application both the wave length of the radio waves and the distances over which it is observed are very small compared to the radius of the earth. He, therefore, introduces a system of curvilinear coordinates which in the first approximation "flattens" the earth into a plane. The system of coordinates is defined as follows: Let R, θ, φ be spherical polar coordinates with origin at the center of the earth. The new coordinates are x, y, z (not rectangular!) defined by the equations: $x=a\theta\cos\varphi$, $y=a\theta\sin\varphi$, $z=a\log(R/a)$, where a is the radius of the earth. In terms of these coordinates the metric ds^2 is $\eta^2(dx^2+dy^2+dz^2)+$terms of order $1/a^2$ where $\eta=\exp(z/a)$. Since the metric does not depend explicitly on x and y, Maxwell's equations in these new coordinates can be solved by Fourier integrals. The resulting solution is essentially equivalent to that obtained by G. N. Watson. [For tables see Domb, Advances in Physics **2**, 96–102 (1953); these Rev. **14**, 691.] *B. Friedman.*

Schumann, W. O. Über die Ausbreitung sehr langer elektrischer Wellen um die Erde und die Signale des Blitzes. Nuovo Cimento (9) **9**, 1116–1138 (1952).

The problem reduces to solving the equation $\Delta u+k^2u=0$ in the region bounded by the spheres $r=a$ and $r=b$. There is a dipole at $\theta=0$, $r=r_0$, $a<r_0<b$, and u satisfies the boundary conditions $(ru)_r=0$ at $r=a$ and $r=b$. Instead of expanding u in spherical harmonics, the author uses an expansion in terms of eigenfunctions of the radially separated part of the wave equation. These eigenfunctions are obtained by finding the values of λ for which there exist non-zero solutions of $(r^2y')'+(k^2r^2-\lambda)y=0$ such that $(ry)'=0$ for $r=a$ and $r=b$. The author computes a few eigenvalues and then discusses the physical implications of the solution. *B. Friedman* (New York, N. Y.).

Lacey, L. J. The mutual impedance of earth-return circuits. Proc. Inst. Elec. Engrs. Part IV. **99**, 156–167 (1952).

Using formulas developed by Sommerfeld [Ann. Physik. (4) **81**, 1135–1153 (1926)] for the electromagnetic fields in the earth and in the air due to horizontal and vertical dipoles, Foster derived an integral expression [Bell System Tech. J. **10**, 408–419 (1931)] for the mutual impedance between two insulated wires lying on the surface of earth, valid for low frequencies, i.e., negligible displacement current. The author of the present paper evaluates this integral in terms of functions for which he provides rather extensive tables. *C. H. Papas* (Pasadena, Calif.).

Moullin, E. B., and Phillips, F. M. On the current induced in a conducting ribbon by the incidence of a plane electromagnetic wave. Proc. Inst. Elec. Engrs. Part IV. **99**, 137–150 (1952).

The current in a perfectly conducting ribbon, i.e., a flat elliptic cylinder, induced by an incident plane wave polarized parallel to the ribbon is investigated. By an extension of Morse and Rubinstein's analysis [Physical Rev. (2) **54**, 895–898 (1938)] a series expansion in terms of Mathieu functions is deduced for the current. This expansion is numerically evaluated and extensive curves which demonstrate the current distribution across ribbons of width $2\lambda/\pi$ and $4\lambda/\pi$ (λ=wavelength) for various angles of incidence are displayed. A comparison of the currents induced at normal incidence near the edge of a half-plane and near the edge of a ribbon indicates that the edge effect is essentially independent of the width of the ribbon, provided the width is at least $\lambda/2$. The diffraction pattern determined from a knowledge of the current appears to be incompatible with Morse and Rubinstein's. This discrepancy is probably due to an inadequate numerical evaluation of the series and not to any theoretical inconsistency. This paper throws considerable light on edge effect phenomena. *C. H. Papas.*

*Brillouin, L. **Wave propagation in periodic structures. Electric filters and crystal lattices.** 2d ed. Dover Publications, Inc., New York, N. Y., 1953. xii+255 pp. Clothbound $3.50; paperbound $1.75.

Reprinted with corrections and additions from the 1st edition [McGraw-Hill, New York, 1946; these Rev. **8**, 422; **9**, 735].

Liebmann, G. The solution of waveguide and cavity-resonator problems with the resistance-network analogue. Proc. Inst. Elec. Engrs. Part IV. **99**, 260–272 (1952).

Lewin, L. The impedance of unsymmetrical strips in rectangular waveguides. Proc. Inst. Elec. Engrs. Part IV. **99**, 168–176 (1952).

The author, following Schwinger's integral equation method, calculates the susceptance of an off-centered capacitive strip and the reactance of an off-centered inductive strip. The strips span a rectangular guide that allows only the dominant mode to propagate. Using the boundary condition that the tangential component of the total (incident+scattered) electric field vanishes at the surface of the strip, he obtains an integral equation for the current induced in the strip. The kernel of the integral equation, which is actually the Green's function for the guide, he replaces by its quasi-static counterpart. He solves this reduced integral equation for the current. The resulting expression for the susceptance, as for the reactance, involves standard elliptic integrals. *C. H. Papas.*

Beck, F. Die Allgemeingültigkeit des Trägheitsgesetzes der Energie in der Planckschen Fassung. Z. Physik **134**, 136–155 (1953).

Minkowski's expression for the energy-momentum tensor of an electromagnetic field interacting with a polarizable medium is not symmetrical. It is shown here that if account be taken also of the matter field tensor, this lack of symmetry does not conflict with angular momentum conservation.

H. C. Corben (Pittsburgh, Pa.).

Marx, G. Relativistische Elektrodynamik der Magnete. Acta Phys. Acad. Sci. Hungar. **2**, 67–84 (1952). (Russian summary)

A theory of the electromagnetic field is developed in which there are permanent magnetic dipoles as well as charged particles. The presence of these magnetic dipoles is taken into account by introducing the additional term $-\frac{1}{2}F_{ik}M^{ik}$ into the Lagrangian, where F_{ik} is the usual antisymmetric tensor and M_{ik} the magnetic moment tensor. The energy-momentum tensor is derived, from which the expression for the force on the magnet is obtained.

C. Kikuchi (East Lansing, Mich.).

Silin, V. P. Superconducting cylinder and sphere in a magnetic field. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **21**, 1330–1336 (1951). (Russian)

Viene applicata la teoria della superconduttività di V. L. Ginsburg e L. D. Landau [stesso Žurnal **20**, 1064–1082 (1950)] al caso di un cilindro e di una sfera superconduttori, immersi in un campo magnetico. Viene calcolata la suscettività magnetica in funzione del campo e della temperatura. Viene studiata la distruzione della superconduttività nei conduttori di piccole dimensioni.

G. Toraldo di Francia (Firenze).

Saltzer, Charles. The second fundamental theorem of electrical networks. Quart. Appl. Math. **11**, 119–123 (1953).

The author proves the basic existence theorem of electrical networks in the frame-work of the Norton representation, and discusses the relationships between dual representations of networks. *R. Bott* (Ann Arbor, Mich.).

Vogel, Théodore. Réponse d'un quadripôle à des signaux discontinus. Le phénomène de Gibbs et les modes de sommation généralisée. Ann. Télécommun. **7**, 421–428 (1952).

Some of the results concerning the Gibbs phenomena are reviewed. The physical interpretation is somewhat as follows: Let the input to an ideal low-pass filter be a square wave. The output is a partial sum of the Fourier series of the wave and the Gibbs phenomenon is to be expected. By introducing a suitable additional amount of attenuation the output is a partial Cesàro sum of order r. According to Gronwall [Ann. of Math. (2) **31**, 233–240 (1930)] the value $r=.439$ is the smallest value of r for which the Gibbs phenomenon does not occur. Output curves are drawn for $r=0$, .44, and 1 corresponding to ten terms. The curve for $r=.44$ "appears" to have the greatest fidelity.

R. J. Duffin (Pittsburgh, Pa.).

*Gibbs, W. J. **Tensors in electrical machine theory.** Chapman & Hall, Ltd., London, 1952. xii+238 pp. 30 shillings.

While tensors have been applied to particle dynamics and to field problems only by professional mathematicians, the use of tensors for the study of circuit problems and in general for the study of engineering structures has been initiated and pursued by the engineers themselves. It took an engineer to be awed by the tremendous complexity of his physical systems on the one hand and to sense the unlimited potentialities of the tensorial (invariant) reasoning for the solution of just such intricate problems on the other. As always in the past, the engineer with an idea and a goal had to do his own mathematical experimentation in a territory where no mathematician had trodded before. The present book, written by an engineer also with a vision, retraces some of the steps that had to be covered in the attempt to organize the physical structures and to unify in a truly dynamical manner—and not just by some trick theory—the analysis of the large variety of rotating electrical machinery used in industry.

The book presents in a satisfactory manner (satisfactory to the engineer at least) the basic topological concept of tearing a group of windings apart into their ultimate components and rotating them into a magnetically interlinked set of two-dimensional concentric networks, the so-called "primitive" rotating machine. The equation of performance of every practical type of rotating machine under all operating behaviour may be derived from that of the primitive machine by a singular transformation.

The mathematical concepts appearing in the book should be ideal targets for mathematicians to shoot at. The concept to consider the tearing apart of a network as a transformation of reference frames is quite unorthodox; so also is the introduction of torsion tensors into classical dynamical problems. The appearance of such recently developed geometrical concepts as non-Riemannian spaces rigged up with non-holonomic reference frames also should not go undisputed. Electric generators and electric motors will be around for some time to come and their analysis offers many

challenges to the topologist, as well as to the differential analyst. *G. Kron* (Schenectady, N. Y.).

Vowels, R. E. Transient analysis of synchronous machines. Proc. Inst. Elec. Engrs. Part IV. **99**, 204–216 (1952).

The paper reviews the transient theory of synchronous machines and sets out the two-reaction transformation in matrix notation. The advantages of the matrix-transformation theory and the Laplace-transform method are demonstrated. *From the author's summary.*

Quantum Mechanics

Takabayasi, Takehiko. On the formulation of quantum mechanics associated with classical pictures. Progress Theoret. Physics **8**, 143–182 (1952).

Following Bohm [Physical Rev. (2) **85**, 166–179, 180–193 (1952); these Rev. **13**, 709, 710] the Schrödinger equation is split into two equations. One is similar to the classical Hamilton-Jacobi equation. The other is a conservation equation. Interpretational difficulties which arise when trying to understand these equations in terms of an ensemble of classical particles are discussed. In particular, difficulties with spinning particles obeying the Pauli principle are strongly emphasized. However, various possible modifications of quantum mechanics are suggested on the basis of this splitting of equations and the utility of the splitting for some special problems is shown.
K. M. Case (Ann Arbor, Mich.).

Fubini, S. Sull'operatore $U(t)$ di Dyson-Feynman. Nuovo Cimento (9) **9**, 846–851 (1952).

This paper gives a brief and clear account of the logic of the process of "diagonalizing the Hamiltonian" in quantum theory, concerning which there has been in the past much confusion of thought. Let the total Hamiltonian of a system be $H=E+V$, where both E and V are independent of time, E represents free particle energy and V interaction energy. The "time-dependent" interaction energy is defined by $V(t)=\exp(iEt)\ V\exp(-iEt)$. The transformation operator $U(t)$ is defined as follows. Let $\lambda(\alpha, t)$ be a function of time and of the real parameter α, satisfying $\lambda(\alpha, t)\to 1$, $\partial\lambda(\alpha, t)/\partial t\to 0$ as $\alpha\to 0$ for every finite t, while $\lambda(\alpha, t)\to 0$ as $t\to-\infty$ for every $\alpha\neq 0$. Then $U(\alpha, t)$ is defined by the differential equation $dU(\alpha, t)/dt+i\lambda(\alpha, t)V(t)U(\alpha, t)=0$ with the initial condition $U(\alpha, -\infty)=1$. Finally, we define $U(t)$ for every finite t by $U(t)=\lim_{\alpha\to 0}U(\alpha, t)$. The time-independent form of $U(t)$ is W defined by $W=\lim_{\alpha\to 0}W(\alpha, t)$, $W(\alpha, t)=\exp(-iEt)\ U(\alpha, t)\exp(iEt)$.

Now it can be proved rather easily that $HW=WE$ and $\bar{W}W=1$. Thus if Φ_a is an eigenstate of E with eigenvalue E_a, then $HW\Phi_a=E_aW\Phi_a$, and thus $W\Phi_a$ is an eigenstate of H with the same eigenvalue. This is what one means by calling W the operator which "diagonalizes the Hamiltonian". It generates an eigenstate of the total Hamiltonian from every eigenstate of the free-particle Hamiltonian E. Further, the states $W\Phi_a$ are an orthonormal set if the Φ_a are orthonormal.

The interesting circumstance is that W is in general not a unitary operator. Thus we write $W\bar{W}=\mathcal{S}$ and $\mathcal{S}$ satisfies $\mathcal{S}^2=\mathcal{S}$, but is in general not the identity operator. The $W\Phi_a$ are just the simultaneous eigenstates of H and $\mathcal{S}$ for which $\mathcal{S}$ has the eigenvalue 1; they do not form a complete set unless $\mathcal{S}=1$. The remaining eigenstates of H are eigenstates of $\mathcal{S}$ with eigenvalue 0, and are not of the form $W\Phi_a$. These eigenstates are the discrete states of the system H and do not have any analog in the free-particle system E.
F. J. Dyson (Ithaca, N. Y.).

Miyatake, Osamu. On the non-existence of solution of field equations in quantum mechanics. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. **2**, 89–99 (1952).

Let H be the Hamiltonian operator for a quantum-mechanical system consisting of a single nucleon interacting with a scalar meson field. The interaction is as usual taken to be strictly a point interaction. The author first defines rigorously the Hilbert space which is generated by those states of the system in which the nucleon has a definite momentum and a finite number of mesons are present with definite momenta. This Hilbert space gives a satisfactory description of the possible states of a free nucleon and a meson field without interaction. The author then proves that the Hamiltonian H with interaction is not an operator of this Hilbert space. In fact, H operating on any state consisting of a definite number of particles gives a vector outside the Hilbert space. From this the author concludes that solutions of the field equations do not exist. The reviewer feels this conclusion to be unjustified; all that has been proved is that the Hilbert space which describes a noninteracting system is inappropriate for the description of a system with point interactions. *F. J. Dyson.*

Rayski, Jerzy. On non-local quantum electrodynamics. Acta Phys. Polonica **11**, 109–130 (1952).

The author constructs a formal theory of the quantized Maxwell and Dirac fields with interaction, embodying the principle of reciprocity of Born [Nature **163**, 207–208 (1949)] and using nonlocal fields as defined by Yukawa [Physical Rev. (2) **77**, 219–226 (1950); these Rev. **11**, 567]. The starting equations are (i) a set of integral equations defining the interacting fields in terms of "ingoing" fields without interaction, (ii) a set of commutation rules for the ingoing fields. This follows the treatment of ordinary quantum electrodynamics by Yang and Feldman [Physical Rev. (2) **79**, 972–978 (1950); these Rev. **12**, 569]. The author claims that a divergence-free S-matrix can be obtained from the integral equations of motion, again following the method of Yang and Feldman. From this S-matrix the physical consequences of the formalism could in principle be discovered.

In the reviewer's opinion a theory of this kind is not physically acceptable unless it can be shown that the interacting fields commute with each other at two points separated by a space-like interval large compared with atomic dimensions. This question is not discussed by the author. It is not clear to the reviewer that the Yang-Feldman construction will lead to interacting fields obeying the principle of causality over large time-intervals.
F. J. Dyson (Ithaca, N. Y.).

Rayski, J. On a regular field theory. Nuovo Cimento (9) **10**, 1–26 (1953).

This is an exposition of the author's field theory with nonlocal interactions, of which he has published several versions [Proc. Phys. Soc. Sect. A. **64**, 957–968 (1951); Philos. Mag. (7) **42**, 1289–1297 (1951); these Rev. **13**, 609, 1010]. In this paper he lays stress on the idea that the integral equations defining his field-quantities should involve integrations only over a finite region Ω of space-time. Then, for given initial values on the boundary of Ω, the

equations will have finite regular solutions over the interior of Ω if the coupling constant is sufficiently small (depending on Ω). The divergence difficulties of ordinary field theory are thus completely absent. However, the physical behavior of the solutions of the equations is not investigated.

F. J. Dyson (Ithaca, N. Y.).

Rumer, Yu. B. Action as a space coordinate. VII. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **23**, 35–48 (1952). (Russian)

This paper is the seventh of a series [for part VI see same Žurnal **22**, 742–754 (1952); these Rev. **14**, 606] concerned with the development of a 5-dimensional generally covariant quantum theory. The theory is here applied to derive the wave-equation for the electron interacting with an external electromagnetic field. The wave-equation is derived by applying the principle of stationary action, equating to zero the variation of the Lagrangian integrated over the 5-dimensional space. The resulting equation is

$$[-ih(\partial/\partial x_k)+(e/c)A_k]\gamma_k\psi-imc\psi+(eh/8mc^2)F_{ik}\gamma_i\gamma_k\psi=0.$$

It differs from the usual Dirac equation by the presence of the last term, which is a direct Pauli-type interaction between the electron spin and the electromagnetic field-strengths F_{ik}. The author says nothing about the physical consequences of this equation. The reviewer observes that the additional term represents an anomalous magnetic moment equal to minus one half of the ordinary Dirac moment of the electron. Therefore in the author's theory the electron will have a gyromagnetic ratio of 1 instead of the observed value 2. This makes the theory quite inadmissable as a theory of real physical electrons.

F. J. Dyson (Ithaca, N. Y.).

Salam, Abdus. Renormalized *S*-matrix for scalar electrodynamics. Physical Rev. (2) **86**, 731–744 (1952).

In this paper the author proves that the consistent use of the idea of renormalization of constants succeeds in removing all the divergences from perturbation calculations to any order in the fine-structure constant α, in the quantum electrodynamics of charged particles with spin zero. The proof is much more difficult than for the case of particles with spin 1/2 which was considered by the reviewer [Physical Rev. (2) **75**, 1736–1755 (1949); these Rev. **11**, 145]. The extra difficulty mainly arises from the additional interaction term $e^2A_\mu^2$ which is present in the scalar theory and which enormously increases the variety of perturbation theory processes which must be considered. Another separate source of new difficulties is the fact that in the scalar theory the scattering of one meson by another gives a divergence in order α^2. It is impossible here to summarise the author's arguments which are based on a detailed combinatorial analysis of the Feynman diagrams to which the theory gives rise.

Two important conclusions emerge from this work. First, that the renormalization method succeeds in this very difficult case, which goes far beyond the theory which the method was originally designed to handle, and where a failure would have been disastrous because it is known experimentally that charged particles with spin zero exist. Second, that the method does not succeed without introducing a new fundamental constant λ in addition to the constants m (mass) and e (charge) which appear in spin-1/2 electrodynamics. All three constants m, e, λ must be renormalized simultaneously. The λ is the coefficient of a fundamental interaction $\lambda(\phi^*\phi)^2$ between two scalar particles. It is remarkable that this one additional constant is also required in order to renormalize the theory of pseudoscalar mesons interacting with a nucleon field [Salam, ibid. **82**, 217–227 (1951); these Rev. **13**, 607]. Further, the one constant is sufficient to renormalize the complete theory of charged pseudoscalar mesons and charged nucleons in interaction with each other and with the electromagnetic field. It thus appears that there is a close correspondence between the theories which can be renormalized and the theories which qualitatively describe the kinds of particles which we observe in the real world. *F. J. Dyson* (Ithaca, N. Y.).

Lévy, Maurice M. Non-adiabatic treatment of the relativistic two-body problem. Physical Rev. (2) **88**, 72–82 (1952).

This paper represents a very substantial advance in our techniques for applying relativistic quantum theory to actual physical problems. Previously, the only problems which could be satisfactorily handled were one-body problems, or scattering problems in which the particle interactions could be regarded as weak. The author here develops a practical method for handling the two-body problem, restricted only by the requirement that the particle velocities be non-relativistic. The method has been applied successfully by the author [in a later paper] to the proton-neutron system at low energies, in particular, to explain the properties of the deuteron.

Previous methods of handling the relativistic two-body problem were based on the adiabatic approximation. The "potential" between the two particles was calculated by finding the energy of the total system, including the meson field, assuming the two particles to be stationary and fixed at a given distance. The equations of motion of the particles was then taken to be an ordinary Schrödinger equation with this adiabatic potential put in as the interaction energy. It has long been clear that this procedure was unreliable, since it cannot take velocity-dependent forces into account at all.

A correct method should derive the interaction energy and the equations of motion together and not separately. Such a method was invented by I. Tamm [Acad. Sci. USSR J. Phys. **9**, 449–460 (1945)] and S. M. Dancoff [Physical Rev. (2) **78**, 382–385 (1950)]. Tamm and Dancoff applied their method incorrectly to the system of two particles interacting through a scalar meson field, since they did not take into account the effects of two mesons existing in the field simultaneously. The main content of the present paper is to show how the Tamm-Dancoff method should be organized and consistently applied in order to obtain velocity-dependent forces correctly. As an illustration, the forces in the two-particle system with scalar meson field are calculated to order $(v/c)^2$.

A second non-adiabatic method for the two-body system has in principle existed for some time, namely the relativistic wave-equation of Salpeter and Bethe [see the following review]. In practice, however, the Bethe-Salpeter equation has not been useful since there is no known method of solving it. The second big contribution of the paper under review is to provide a systematic approximation method for solving the Bethe-Salpeter equation, in which the adiabatic solution is taken as the zero-order approximation and successive iterations of the equation yield successive non-adiabatic approximations to the true solution. The author shows that his method of approximating to the Bethe-Salpeter equation gives results identical with the

Tamm-Dancoff method as long as the particles are restricted to positive-energy states. However, in a relativistic theory negative-energy states also must be taken into account, and this introduces small differences between the Bethe-Salpeter and Tamm-Dancoff results. It is not yet clear which of the two methods is to be considered the more accurate. The author's treatment of the negative energy states is based on the ideas of Tamm and Dancoff, but in its details is quite new. *F. J. Dyson* (Ithaca, N. Y.).

Salpeter, E. E., and Bethe, H. A. A relativistic equation for bound-state problems. Physical Rev. (2) **84**, 1232–1242 (1951).

The Feynman formalism [Physical Rev. (2) **76**, 749–759, 769–789 (1949); these Rev. **11**, 765] is extended to bound state problems involving two Fermi-Dirac particles which interact with each other by means of an arbitrary interaction in the absence of any external forces, particles or quanta. First, an integral equation for the two-particle wave function $\psi(1, 2)$ is derived; its kernel consists of a product of the Feynman amplitude functions for the two particles and an interaction function G which is a sum of expressions $G^{(n)}$ corresponding to all the irreducible Feynman graphs. No closed expression for G has as yet been found. If the coupling constant is reasonably small, considerable simplification is gained from a treatment involving summation over all reducible graphs. Further simplification is achieved by using the total and relative momenta as variables. This leads to an integral equation for a momentum space wave function depending on only four variables, namely, the components of the relative momentum of the two interacting particles, and their "relative energy". Application of the product of Feynman operators $(i\nabla_1^a - m_a)\cdot(i\nabla_2^b - m_b)$ to the integral equation transforms it into the integral-differential equation

$$(i\nabla_1^a - m_a)(i\nabla_2^b - m_b)\psi(1, 2) = i\int\int d\tau_3 d\tau_4 G(1, 2; 3, 4)\psi(3, 4)$$

which is the analog of the Dirac equation for a Fermi-Dirac particle in an external field (1, 3 and 2, 4 refer to time and place of particle a and b; $d\tau$ is the four-dimensional volume element). This equation is more convenient for practical use than the integral equation; however, only those solutions of the integral-differential equation are physically acceptable which can be shown to satisfy the integral equation. From a practical point of view it would be more satisfactory to have some criteria of "good behavior" which would determine directly which of the solutions of the integral-differential equation correspond to physical states, without any reference to the integral equation, but no completely satisfactory set of such criteria has as yet been found. It is suggested that a fictitious adiabatic variation of the coupling constant may be of some use in a search for such criteria. The integral-differential equation is applied to the deuteron ground state using a scalar meson theory with scalar interaction. *E. Gora* (Providence, R. I.).

Hamilton, J. Steady states and the *S*-matrix. Proc. Cambridge Philos. Soc. **49**, 97–102 (1953).

A relation between Salpeter-Bethe's equation [see the preceding review] and Dyson's S-matrix [Physical Rev. (2) **75**, 1736–1755 (1949); these Rev. **11**, 145] is pointed out and discussed. First, the S-matrix formalism is used to derive an equation for a steady state function Φ; this function defines an ensemble of free two-particle states which, when the interaction is adiabatically switched on, give rise to a bound state or to a state describing elastic scattering. To solve this equation, an auxiliary function $\Psi(k_1, k_2)$ of the energies and momenta k_1, k_2 of the two fermions is introduced; it is defined as "the sum of the effect of all partially completed two-body graphs acting on a steady state function Φ". For $\Theta = \Phi + \Psi$ an equation is then obtained which is identical with the Bethe-Salpeter equation. Capture and emission of bosons is discussed. *E. Gora.*

Galanin, A. D. A relativistic equation of interacting particles. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **23**, 488–492 (1952). (Russian)

The author has previously [Doklady Akad. Nauk SSSR (N.S.) **79**, 229–232 (1951); same Žurnal **22**, 448–461, 462–470 (1952); these Rev. **13**, 413; **14**, 436] developed a method of evaluating radiative corrections in quantum electrodynamics by successive approximations. He now applies this method to the problem of two interacting fermions; and obtains a relativistic equation of the form:

$$(p_1 - m)(p_2 - m)\psi(x_1, x_2) = V_{12}\psi(x_1, x_2)$$

where p_1 and p_2 are the energy-momentum operators of the two particles. The operator V_{12} is worked out explicitly in the approximations where the participation of one and two photons is taken into consideration. Transitions represented by reducible Feynman diagrams are omitted. The results obtained agree with the corresponding results of Bethe and Salpeter [see the second preceding review]. *E. Gora.*

Günther, Marian. The relativistic configuration space formulation of the multi-electron problem. Physical Rev. (2) **88**, 1411–1421 (1952).

The Tomonaga-Schwinger formalism and the Dirac-Fock-Podolski many-time theory [Tomonaga, Progress Theoret. Physics **1**, 27–42 (1946); these Rev. **10**, 226; Schwinger, Physical Rev. (2) **74**, 1439–1461 (1948); these Rev. **10**, 345; Dirac, Fock, and Podolski, Physik. Z. Sowjetunion **2**, 468–479 (1932); referred to as T-Sch. and D.F.P. respectively] are shown to be equivalent for many-electron systems, provided that the latter is supplemented by the Fermi-Dirac statistics. First, it is proved that functions obeying the D.F.P. equations can be expanded in a series whose coefficients depend functionally on space-like hypersurfaces. These coefficients are used to construct the T-Sch. state vector. The T-Sch. equation of motion for the state vector is then obtained by taking into account the Fermi-Dirac statistics. Inversely, it is shown that the components of every solution of the T-Sch. equations represent expansion coefficients of functions obeying the D.F.P. equations. The proposed formalism permits a formal generalization which covers all the points of $4n$-dimensional configuration space while the original D.F.P. formalism is defined only for space-like particle distances; it can now be directly applied to bound-state problems of many-electron systems. Finally, as starting point for the actual calculations an integral equation and integro-differential equations are derived. They are worked out explicitly in the e^2-approximation. A method of eliminating virtual processes is outlined, and invariant expressions for the many-electron interactions are derived. For the two-electron problem an equation is obtained which is identical with the corresponding equation of Bethe and Salpeter [see the third preceding review]. *E. Gora* (Providence, R. I.).

Arnous, E., und Bleuler, K. Allgemeine Theorie der Dämpfungsphänomene für nichtstationäre Prozesse. II. Abseparierung der virtuellen Zustände. Korrektionen zweiter Ordnung. Helvetica Phys. Acta **25**, 581–598 (1952).

In part I of this paper [Arnous and Zienau, same Acta **24**, 279–295 (1951); these Rev. **13**, 195] a formalism for the treatment of bound-state problems has been developed which is based on an extension of the S-matrix formalism and permits one to take into consideration damping effects. This formalism is now used to calculate higher order corrections to the form of emission lines. A fully consistent use of the formalism would require an unambiguous way of separating real and virtual states and processes. At present this can be done only in a somewhat arbitrary way. The authors introduce for this purpose "finite energy shells" of width ϵ, and define states outside these shells as virtual. For the ground state, naturally, $\epsilon(G)=0$; the arbitrariness in the choice of ϵ for the other states is linked up with the fact that they can be excited in different ways. All the operators are then split up into two parts, one with its matrix elements inside, the other with its matrix elements outside the energy shell. A transformation operator, which achieves this, is determined by successive approximation, and applied to the renormalized Hamilton function. This leads to an operator function K. The radiative corrections to the line form are determined by applying the formalism of part I to K. It is proved that the displacement of the line maximum is almost exactly given by the level shifts as usually calculated, and does not depend upon ϵ. The corrections are worked out explicitly in the e^2-approximation. The line width correction depends upon ϵ in this approximation and vanishes for $\epsilon\to\infty$. *E. Gora* (Providence, R. I.).

Arnous, E. Allgemeine Theorie der Dämpfungsphänomene für nichtstationäre Prozesse. III. e^4-Korrekturen zur Linienbreite. Helvetica Phys. Acta **25**, 631–652 (1952).

The methods developed in parts I and II [see the preceding review] are used to determine the e^4-correction to the line width. It is shown that renormalization of charge and mass is sufficient to remove all the divergencies from the matrix elements, and that the result is unambiguous apart from its dependence upon the arbitrary width of the energy shell ϵ [see the preceding review]. Methods of evaluating the matrix elements are formulated in a covariant way, but an estimate of the order of magnitude of the physically interesting results is given in the non-relativistic approximation only. The result consists of two parts: one part is of the order of magnitude $\gamma/137^2$ where γ is the line width; the other part is of the order of magnitude γ^2/ϵ. A more precise formulation of the introduction of the width of the energy shell ϵ would seem desirable, but, according to the author, this "has nothing to do with the typically quantum electrodynamical aspects of the problem". *E. Gora.*

Hori, Shoichi. On the well-ordered S-matrix. Progress Theoret. Physics **7**, 578–584 (1952).

The author obtains a formal closed expression for the perturbation theory expansion of the S-matrix in quantum electrodynamics. This is

$$S=:\exp(\Sigma)\exp(\Delta)\exp\left[i\int_{-\infty}^{\infty}j_\mu(x)A_\mu(x)dx\right]:$$

where the : : notation means that all operators within the colons are to be written with creation operator factors standing to the left of annihilation operator factors. Δ is the operator of functional differentiation

$$\Delta=\frac{1}{2}\int_{-\infty}^{\infty}\int D_F(x-y)[\delta^2/\delta A_\mu(x)\delta A_\mu(y)]dxdy,$$

and similarly

$$\Sigma=\frac{1}{2}\int_{-\infty}^{\infty}\int S_{F\alpha\beta}(x-y)[\delta^2/\delta\psi_\alpha(x)\delta\bar{\psi}_\beta(y)]dxdy.$$

As the author remarks, these formulae are closely related to Feynman's operator calculus [R. P. Feynman, Physical Rev. (2) **84**, 108–128 (1950); these Rev. **13**, 410]. They do not make any change in the practical use of the theory, since it seems that operations $\exp(\Sigma)$ and $\exp(\Delta)$ can only be defined in terms of a power series expansion of the exponentials. *F. J. Dyson* (Ithaca, N. Y.).

Thirring, Walter. On the divergence of perturbation theory for quantized fields. Helvetica Phys. Acta **26**, 33–52 (1953).

The author considers the quantum field theory of a single scalar field ψ representing particles interacting with each other in virtue of a term $\lambda\psi^3$ in the Lagrangian. He considers the power series in λ obtained by applying perturbation theory to this system and using renormalization in order to remove divergence arising in each order separately. He proves that this series after renormalization diverges, the nth term being positive and of the order of magnitude $(C\lambda)^n(\frac{1}{2}n)!$ where C is a numerical constant. This result has been previously published by C. A. Hurst [Proc. Cambridge Philos. Soc. **48**, 625–639 (1952); these Rev. **14**, 607] with a proof which was admitted to be incomplete. The proof in the present paper is complete and was for the most part derived independently, although the published version contains some simplifications resulting from Hurst's work. The proof consists of a straightforward but delicate estimation of the magnitude of the Feynman integrals occurring in the theory, combined with a verification that all contributions to each term of the series are separately positive. The improvement over Hurst's method lies in a systematic treatment of the finite residues remaining after renormalization of self-energy processes, which Hurst omitted from his analysis. *F. J. Dyson* (Ithaca, N. Y.).

Fabre de la Ripelle, Michel. Résolution des équations de perturbation. II. Forme des probabilités. C. R. Acad. Sci. Paris **236**, 459–461 (1953).

[For part I see same C. R. **234**, 412–414 (1952); these Rev. **13**, 805.] The probability of finding a system in a certain state after a certain number of transitions from a given initial state is computed by probability theory and conditions are deduced in order that the form of this probability function should agree with that computed from perturbation theory. *H. C. Corben* (Pittsburgh, Pa.).

Rzewuski, J. On the interaction of particles in Feynman's theory. Acta Phys. Polonica **10**, 294–296 (1951).

The lowest order modifications of nucleon propagation functions due to a charged vector meson field is obtained. This is done by first postulating the form of the meson term in the nucleon equations of motion. Ascribing the meson field to a current of unknown form caused by a second nucleon and solving perturbation-wise gives the transition amplitude for one nucleon caused by a transition of a second nucleon.

Equating this to the amplitude corresponding to interchanged nucleons yields the form of the "current" and thence the modified propagation function. The direct method by means of the Lagrangian seems to be avoided. *K. M. Case* (Ann Arbor, Mich.).

Rzewuski, Jan. Statistical interpretation of the Klein-Gordon equation. Acta Phys. Polonica **10**, 296–299 (1951).

A space-time approach [cf. Feynman, Physical Rev. (2) **76**, 749–759 (1949)] is applied to particles satisfying the Klein-Gordon equation. *K. M. Case.*

Rzewuski, Jan. A note on perturbation theory. Acta Phys. Polonica **11**, 179–188 (1952).

The identity of the results of the old and new perturbation methods used in field theory [cf. Heitler, The quantum theory of radiation, 2nd ed., Oxford, 1944; Feynman, Physical Rev. (2) **76**, 749–759, 769–789 (1949); these Rev. **11**, 765] is shown by transition from the formulae obtained by one method to that of the other in three simple cases. It is shown that the old results are obtained from Feynman's method by: (a) decomposing the kernels into sums over eigenstates; and (b) performing the time integrations first. *K. M. Case* (Ann Arbor, Mich.).

Takahashi, Yasushi, and Umezawa, Hiroomi. On the self-stress. Progress Theoret. Physics **8**, 193–204 (1952).

To treat the self-stress problem the following generalization of the Pais-Epstein formula [Pais and Epstein, Rev. Modern Physics **21**, 445–446 (1949)]

$$\left\langle \int T_{uu} dv \right\rangle = u\frac{\partial}{\partial u}\delta u + x\frac{\partial}{\partial x}\delta u$$

is shown to be true to all orders in perturbation theory for any two interacting fields. Combining this with general dimensional considerations, the author shows the self-stress to vanish to all orders. The difficulties previously encountered when employing cut-off procedures is shown to be due to the introduction of a quantity with the dimensions of length. *K. M. Case* (Ann Arbor, Mich.).

Umezawa, Hiroomi. On the structure of the interactions of the elementary particles. II. Does the interaction of the second kind exist in the nature? Progress Theoret. Physics **7**, 551–562 (1952).

Continuing a discussion begun in an earlier paper [S. Sakata, H. Umezawa, and S. Kamefuchi, same journal **7**, 377–390 (1952); these Rev. **14**, 608], the author here makes a detailed list of the different possible combinations of fields and interactions, classifying the interactions into the two kinds defined previously. For the renormalization method to succeed in giving us a divergence-free theory of elementary particles, it is necessary and sufficient that all interactions occurring in nature be of the first kind. Going through his list, the author shows that all the interactions for which good experimental evidence exists are of the first kind, with one exception. The exception is the direct coupling of 4 spinor fields in the generally accepted theory of β-decay. It is possible to remove this exception by supposing the β-decay interaction to be not a direct coupling, but a second-order effect of a coupling of the spinor field to a hypothetical spin-zero field. The spin-zero field must give rise to particles heavier than the π-meson, which may be identical with one of the types of heavy particle already observed. There is thus no conclusive evidence either for or against the existence of interactions of the second kind. Two appendices contain useful results concerning the formal equations of the neutral vector field and the commutation relations of higher spin fields respectively. *F. J. Dyson* (Ithaca, N. Y.).

Lüders, Gerhart. Zur Bewegungsumkehr in quantisierten Feldtheorien. Z. Physik **133**, 325–339 (1952).

"Time inversion" is considered from the viewpoint of a reversed motion of the system, i.e., it is stated in the form that: (a) to every state Ψ there should be a state $\hat{\Psi}$ describing reversed motion such that transition probabilities are the same: $(\Psi, \Phi) = (\hat{\Psi}, \hat{\Phi})^*$; and (b) that if the state Ψ_1 at t_1 go over to Ψ_2 at t_2, then a state $\hat{\Psi}_2$ at t_1 should go over to $\hat{\Psi}_1$ at t_2. It is shown that there is a correspondence of operators and wave functions such that the quantized theory of Dirac electrons interacting with the electromagnetic field do have an invariance of this form. Some consequences of this postulate for general quantized fields are drawn and the possible non-existence of the invariance is considered. *K. M. Case* (Ann Arbor, Mich.).

König, H. W. Materielle und quantenhafte Eigenschaften elektromagnetischer Wellenfelder. Acta Physica Austriaca **4**, 405–430 (1951).

König, H. W. Zu einem elektromagnetischen Wellenbild von Mikrovorgängen. Acta Physica Austriaca **5**, 286–318 (1952).

By an appropriate splitting of the electromagnetic energy density, an interpretation of the electromagnetic field is given. Possible quantum theoretical consequences are discussed. *C. Kikuchi* (East Lansing, Mich.).

Groschwitz, E. Beiträge zur Feldmechanik. Z. Naturforschung **7a**, 458–465 (1952).

The motion of field singularities corresponding to a Bopp-Feynman type of generalized linear electrodynamics is considered. Discussion is restricted to terms in the "particle" Lagrangian which involve at most the first derivative of the velocity. The evaluation of these terms by power series development is shown to lead to convergence difficulties. For a particular field model this part of the Lagrangian is found in closed form. The self-energy and fluctuation motion corresponding to this model are then discussed. *K. M. Case* (Ann Arbor, Mich.).

Wessel, Walter. Zur Theorie des Elektrons. III. Z. Naturforschung **7a**, 583–593 (1952).

An attempt is made to account for such radiation reaction effects as the breadth of quantum states and the Lamb-Retherford shift by a model which avoids quantisation of the radiation field but adopts a more complicated description of the electron. The spin is treated by the methods of Rosen [Physical Rev. (2) **82**, 621–624 (1951); these Rev. **13**, 610]. The paper is mainly mathematical. Certain operators previously used in part II [same Z. **6a**, 478–483 (1951); these Rev. **13**, 807] are reinterpreted. The additional magnetic moment of the electron does not agree with experiment. *C. Strachan* (Aberdeen).

Tokuoka, Zensuke, and Tanaka, Hajime. On the equivalence of the particle formalism and the wave formalism of meson. Progress Theoret. Physics **8**, 599–614 (1952).

The equivalence of the β formalism and the wave formalism for scalar and vector particles interacting with the electromagnetic field is demonstrated by constructing the

S matrix, following Feynman and Dyson. These are shown to be identical despite the computational difference of having only 3-vertices in the first case and 3- and 4-vertices in the second. *K. M. Case* (Ann Arbor, Mich.).

Shanmugadhasan, S. The quantization of classical spin theory. Canadian J. Physics **31**, 1–10 (1953).

In order that the classical theory of spinning particles should describe a particle which has zero electric moment in the rest system it is necessary to impose as a supplementary condition a relation between the velocity and the spin tensor. It is therefore possible to quantize the theory by using the method developed by Dirac for taking into account such a supplementary condition. This quantization is carried out for the cases of a pure gyroscope and a symmetrical top in a c-number electromagnetic field.

H. C. Corben (Pittsburgh, Pa.).

Kynch, G. J. The calculation of scattering amplitudes. Proc. Phys. Soc. Sect. A. **65**, 708–718 (1952).

An integro-differential equation is obtained for the scattering amplitude corresponding to a given incident wave. Since this equation, in essence, involves finding the scattering amplitude when the potential is cut-off at successively larger distances, it should be particularly easy to solve numerically. It is shown that the method is also applicable to ionization problems and can be used for the Dirac equation. *K. M. Case* (Ann Arbor, Mich.).

Haag, Rudolf. Zur korrespondenzmässigen Theorie der Spinwellengleichungen. Z. Naturforschung **7a**, 449–458 (1952).

In the first part of this paper some realizations of the structure relations of the orthogonal group in n dimensions by means of differential operators are discussed. Only single-valued representations are obtained in this way. Secondly, a Hamiltonian formalism is set up for a classical particle whose equations of motion have higher time derivatives than the second. On quantization the "Schrödinger equation" is found to be of the form suggested by many authors for particles of higher spin. However, the commutation relations obtained are just those considered in the first portion and so lead only to particles of integral spin.

K. M. Case (Ann Arbor, Mich.).

Thermodynamics, Statistical Mechanics

Tolhoek, H. A., and de Groot, S. R. A discussion of the first law of thermodynamics for open systems. Physica **18**, 780–790 (1952).

The first law for open systems is a physical law only if there is no diffusion. If there is diffusion, the first law may be formulated in two ways; these formulations represent distinct definitions of heat transfer. One of these definitions is invariant under shifts in energy zero points, but the other is not. Four laws are formulated as a basis for the thermodynamics of irreversible processes in continuous systems.

C. C. Torrance (Monterey, Calif.).

Miller, A. R. The concept of temperature. Amer. J. Phys. **20**, 488–491 (1952).

The author claims to give a mathematical proof that the law of thermal equilibrium leads to the concept of temperature, but he omits essential steps of the reasoning, and the result is unconvincing. *C. C. Torrance.*

Scheidegger, Adrian E., und McKay, Carlyle D. Thermodynamische Vakuumschwankungen der Wellenfelder. Z. Physik **132**, 179–182 (1952).

Krotkov, R. V., and Scheidegger, A. E. Thermodynamics of some special fields. Canadian J. Physics **31**, 329–336 (1953).

Koppe, H. Die Berechnung von Zustandssummen mittels Laplace-Transformationen. Ann. Physik. (6) **9**, 423–428 (1951).

The Slater sum of a quantum mechanical system is expressed by Laplace transform in terms of the Green function of the equation $(H-s)\psi=\psi_0$, where H is the hamiltonian and s a constant. The expression thus obtained is used to derive the asymptotic expansion of the Slater sum in inverse powers of the temperature. The method, the practical applicability of which is restricted to systems with a small number of degrees of freedom, is exemplified for the cases of a one-dimensional particle and a spherical rotator.

L. Van Hove (Princeton, N. J.).

Schönberg, M. Application of second quantization methods to the classical statistical mechanics. I. Nuovo Cimento (9) **9**, 1139–1182 (1952).

It is shown that methods similar to those of second quantization can be applied to the Liouville equation of classical statistical mechanics. The Liouville equation is replaced by a more general "Schrödinger" equation for a wave functional. Properties of operators which can act on this functional are deduced, interpretation rules are postulated, and the connection with conventional formulations are discussed. *K. M. Case* (Ann Arbor, Mich.).

Nishiyama, Toshiyuki. A quantum theory of boson assemblies. I. Progress Theoret. Physics **8**, 655–668 (1952).

A statistical theory of boson assemblies is developed by the method of second quantization, along lines similar to the treatment of many-fermion problems by Tomonaga [same journal **5**, 544–569 (1950); these Rev. **13**, 414]. The operators describing density, current, energy density, stress tensor, and energy flow are derived in terms of the quantized wave function; their commutation rules and the equations of motion are given. A velocity operator is defined and discussed in the limit of large numbers of quanta (classical limit), where a velocity potential is shown to exist and to be canonically conjugate to the density operator.

L. Van Hove (Princeton, N. J.).

Kilpatrick, John E. A quantum-mechanical treatment of virial coefficients. J. Chem. Phys. **21**, 274–278 (1953).

The virial expansion for a quantum gas is obtained by the familiar method of expanding pressure and number density in powers of the activity and eliminating the latter. As an application, the effect of nuclear spin on the second virial coefficient is discussed. In the author's notation, the reviewer finds that α_n is the term independent of z in the Laurent expansion of $n^{-1}(\sum_i^n p_i z^i)^{1-n}$, and not the coefficient of z^{2n}, as stated by the author at the end of the appendix. The direct proof of this fact is elementary and avoids the lengthy proof given in the appendix: put $\varphi=\sum_1^\infty g_i z^i=\sum_1^\infty \alpha_n x^n$, one

has $x=z\partial\varphi/\partial z$,

$$a_n=(2\pi i)^{-1}\oint x^{-n-1}\varphi dx=(2\pi in)^{-1}\oint x^{-n}d\varphi$$
$$=(2\pi in)^{-1}\oint z^{-n}(\partial\varphi/\partial z)^{1-n}dz$$
$$=(2\pi in)^{-1}\oint z^{-1}(z\partial\varphi/\partial z)^{1-n}dz, \text{ q.e.d.}$$

L. Van Hove (Princeton, N. J.).

Reiss, Howard. The statistical mechanical theory of irreversible condensation. I. J. Chem. Phys. **20**, 1216–1227 (1952).

Starting from the Gibbs configuration integral, the author attempts a statistical description of a supersaturated vapor and of its condensation into small liquid drops (called nuclei). The vapor phase is treated by subdividing space into cells. The interaction between particles is taken into account, a low density approximation being used for the interaction between particles located in different cells. The average value $n(x)$ of the number of cells containing x particles is derived. If condensation is to occur, the function $n(x)$, beyond its maximum at the vapor density, has a minimum at a higher value x^* of x; a cell with x^* molecules is defined as a nucleus. The supersaturated character of the vapor is described by cutting off $n(x)$ at a value $x=h$ determined by the condition that the vapor behave like an ideal gas. The consistency of the description requires $h=x^*$; this equation determines the size of the cell, and therefore of the nuclei. From the knowledge of $n(x)$, a formula is derived for the rate of formation of nuclei. One should note that Eqs. (2.14) and (2.36) are approximate since they make no distinction between average potential energy and configurational free energy of a cell. *L. Van Hove.*

Yang, C. N., and Lee, T. D. Statistical theory of equations of state and phase transitions. I. Theory of condensation. Physical Rev. (2) **87**, 404–409 (1952).

General analytical properties are derived for the grand partition function Q_V of a classical system of interacting particles, when the volume V approaches infinity. Q_V is considered as function $Q_V(y)$ of the fugacity y, treated as a complex variable. Under natural assumptions the following properties are proved. For y real and positive and $V\to\infty$, $V^{-1}\log Q_V$ approaches a limit (the pressure divided by kT) which is independent of the shape of V and is a continuous, monotonically increasing function of y. In a region R of the complex y-plane containing a segment of the positive real axis and free of roots of $Q_V(y)=0$ for all V, $V^{-1}\log Q_V(y)$ and its derivatives with respect to $\log y$ approach limits analytic in y, and the operations $\partial/\partial\log y$, $\lim_{V\to\infty}$ commute in R. These results show that phase transitions occur only when $Q_V(y)=0$ has roots which close in onto the real axis for $V\to\infty$ and that the pressure is analytic in y on the real axis between the values of y corresponding to transitions. Below the smallest of these values Mayer's theory of gases is shown to hold. *L. Van Hove* (Princeton, N. J.).

Lee, T. D., and Yang, C. N. Statistical theory of equations of state and phase transitions. II. Lattice gas and Ising model. Physical Rev. (2) **87**, 410–419 (1952).

The general properties of the grand partition function established in the paper reviewed above are applied to the so-called lattice gas model, in which the positions of the particles in the system are restricted to the sites of a lattice, the occupation number of each lattice site taking the values 0 or 1. The lattice gas model is first shown to be equivalent to the Ising model of ferromagnetism with external magnetic field. The results so far known for the two-dimensional Ising model are used to show the existence of a phase transition and a critical temperature for the corresponding two-dimensional lattice gas with attractive potential. That no other transition occurs follows from a remarkable theorem which is then given for all lattice gases with an attractive potential between particles at different lattice sites: the grand partition function, considered as a function of $z=\tau y$, where y is the fugacity and τ a convenient positive constant, has all its roots on the unit circle of the complex z-plane. According to this theorem and the results of the previous paper, a lattice gas with attractive potential has never more than one transition. The distribution of roots on the unit circle is related to the thermodynamic functions, as well as to the coefficients of their Mayer expansion in powers of the fugacity. *L. Van Hove* (Princeton, N. J.).

Rushbrooke, G. S., and Scoins, H. I. On the theory of fluids. Proc. Roy. Soc. London. Ser. A. **216**, 203–218 (1953).

The paper concerns the equilibrium theory of classical fluids. The expansions of pressure p and radial distribution function $g(r)$ in powers of the density ρ are recalled and shown to provide identities when put in the familiar expressions of p and $(\partial\rho/\partial p)_T$ in terms of $g(r)$. According to the Zernike-Prins formula, the scattering intensity function $i(s)$ is introduced as the Fourier transform of $\rho[g(r)-1]$.

The expansion of $[1+i(s)]^{-1}$ in powers of ρ is given; the coefficients are simply related to irreducible cluster integrals. One considers next the direct correlation function $f(r)$ defined by the integral equation

$$g(r)-1=\rho^{-1}f(r)+\int f(|\mathbf{r}-\mathbf{r}'|)[g(r')-1]d\mathbf{r}'$$

and first introduced by Ornstein and Zernike [Nederl. Akad. Wetensch., Proc. **17**, 793–806 (1914)]; the power expansion in ρ is given and it is suggested that $f(r)$ is always short-ranged in r. It is shown that neglecting all terms but one in the expansion of $f(r)$ is essentially equivalent to the crudest form of the Born-Green approximate theory of liquids [M. Born and H. S. Green, A general kinetic theory of liquids, Cambridge Univ. Press, 1949; these Rev. **12**, 230]. A better approximation, essentially obtained by retaining the two first terms of the expansion of $f(r)$ in powers of ρ, is proposed. It has a simple meaning in terms of the cluster integrals retained in the expansion of $g(r)$. It gives correctly the three first virial coefficients and considerably improves the value of the fourth for hard spheres, as compared to previous approximations. The relation with X-ray scattering is also discussed. *L. Van Hove.*

Kubo, Ryogo. Statistical theory of linear polymers. I. Intramolecular statistics. J. Phys. Soc. Japan **2**, 47–50 (1947).

Kubo, Ryogo. Statistical theory of linear polymers. II. Elasticity of vulcanized rubber. J. Phys. Soc. Japan **2**, 51–56 (1947).

Kubo, Ryogo. Statistical theory of linear polymers. III. Double refraction. J. Phys. Soc. Japan **2**, 84–89 (1947).

Kubo, Ryogo. Statistical theory of linear polymers. IV. Effects of hindrance for internal rotation. J. Phys. Soc. Japan **3**, 119–126 (1948).

Kubo, Ryogo. Statistical theory of linear polymers. V. Paraffine-like chain. J. Phys. Soc. Japan **4**, 319–322 (1949).

These papers are concerned with the application of statistical mechanics to the analysis of various properties of

linear polymers. The main mathematical idea is that the partition function of a linear polymer in which each monomer interacts only with its nearest neighbors can be expressed in terms of the largest characteristic value of a certain integral equation. The kernel of this equation is the Boltzmann factor $\exp -\epsilon(kT=K(q, q')$ where ϵ is the interaction energy between a pair of neighboring monomers which are in configurational states q and q'.

E. W. Montroll (College Park, Md.).

Temperley, H. N. V. Statistical mechanics and the partition of numbers. II. The form of crystal surfaces. Proc. Cambridge Philos. Soc. **48**, 683–697 (1952).

[For part I see Proc. Roy. Soc. London. Ser. A. **199**, 361–375 (1949); these Rev. **12**, 231.] A process is given for computing various boundaries for two- and three-dimensional rectangular arrays in thermal equilibrium under the assumption of a surface energy. The number of possible shapes for such an array is associated with the exponential of the entropy. It is shown in various cases that this number can be obtained by means of a generating function given by a power series $\sum a_n x^n$ whose coefficients correspond to various ways of partitioning the index n. The saddle point method is then used to obtain the various equilibrium quantities and relations. Thus the first and simplest result describes the equilibrium shape of a two-dimensional rectangular array located between two perpendicular walls whose surface of contact with the array does not contribute to the surface energy. This also describes the blunted point of a reentrant angle on the boundary of a two-dimensional lattice. Various generalizations in the two-dimensional case are given, including the possibilities for a "sawtooth." The sawtooth question in the three-dimensional case is discussed by reference to P. A. MacMahon's book [Combinatory analysis, v. 2, Cambridge, 1916] and to certain unpublished work of V. Nanda. In both the two- and three-dimensional cases of the "sawtooth," the energy and entropy involve the same power of N, the total number of atoms in the configuration, and hence thermal equilibrium is possible. A complete solution to the two-dimensional boundary case by generating functions, due to Onsager, is also given. This does not employ the saddle point methods.

F. J. Murray (New York, N. Y.).

Berlin, T. H., and Thomsen, J. S. Dipole-dipole interaction in simple lattices. J. Chem. Phys. **20**, 1368–1374 (1952).

An array of dipoles located at the vertices of a rectangular lattice is considered. Ordinarily the direction of dipole is specified by a vector ϵ_j with unit length, $|\epsilon_j|=1$, and the energy between nearest neighbors in the partition function is expressed in terms of these ϵ_j. In this paper, a "spherical model" is considered, which uses the same expression for the partition function except that the restrictions $|\epsilon_j|=1$ are replaced by the single restriction $\sum|\epsilon_j|^2=N$. In the case of a dimensional lattice, the side condition is introduced into the integral expression for the partition function in the form of a δ-function and, after a readjustment necessary to insure convergence, the resulting expression can be evaluated as a definite integral with infinite bounds of the exponential of a quadratic form. The evaluation of the latter integral proceeds in two steps, one of which involves a Stieltjes type determinant, that is, one with constant values along the principle diagonal and two adjacent diagonals, and the latter is evaluated explicitly. The final step in the evaluation of the one-dimensional partition function is a saddle point evaluation of a single integral. It is stated that the two-dimensional case can be handled in a similar way and that neither of these indicates a transition. In the three-dimensional case the Stieltjes type determinant is replaced by a more complicated one which, however, still permits an evaluation by obtaining all the characteristic values. However, in the three-dimensional case the value at which the saddle point occurs is determined implicitly in such a way that a transition is possible with constant specific heat below a certain temperature and decreasing specific heat above. The constant specific heat refers to a region of long range order. It is also shown that the introduction of an electric field decreases the temperature of the transition point and if the field is of sufficient strength, the transition does not occur.

F. J. Murray (New York, N. Y.).

Lax, Melvin. Dipoles on a lattice: the spherical model. J. Chem. Phys. **20**, 1351–1359 (1952).

This paper is an extension of the results of Berlin and Thomsen [see the preceding review] to include a discussion of the interaction of dipoles which are not nearest neighbors. The author points out that the diagonalization of the interaction matrix on which the Berlin and Thomsen result depends requires that the matrix be cyclic which in turn implies a particular shape for the crystal lattice. Shape is not important when only nearest neighbor reactions are involved but becomes significant in the general case. The author claims that the effect of shape is in the limiting process as $N\to\infty$ and that therefore he is justified in replacing the shape limitation by a factor under the integral sign in the expression for the partition function. This modified expression is then considered for the cyclic lattice and a procedure of the same general plan as that of Berlin and Thomsen is followed through. However, the interaction matrix cannot be simplified by considering the x, y, and z directions independently. The one- and two-dimensional cases do not yield a transition. There is a careful discussion of the relation between the saddle point and the greatest characteristic root of the interaction matrix and the effect of this relation on the possibility of a transition.

F. J. Murray (New York, N. Y.).